

ON THE CLASSIFICATION OF NORMAL G -VARIETIES WITH SPHERICAL ORBITS

KEVIN LANGLOIS

ABSTRACT. In this article, we investigate the geometry of reductive group actions on algebraic varieties. Given a connected reductive group G and a normal G -variety X with spherical orbits, we show that X admits a finite Galois covering with total space a G -variety having a trivial G -equivariant birational type (i.e., the flat geometric quotient by G on some G -stable dense open subset is a trivial family). As a consequence, we elaborate a geometrico-combinatorial approach based on the Luna-Vust theory to describe normal G -varieties with spherical orbits. This description comprises the classical case of spherical varieties and the theory of \mathbb{T} -varieties recently introduced by Altmann, Hausen, and Süss.

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INTRODUCTION

We will consider algebraic varieties and algebraic groups over an algebraically closed field k of characteristic 0. For a given connected reductive linear algebraic group G , the purpose of this article is to study the classification of normal G -varieties with spherical orbits. Motivated by the well-known description of toric varieties in term of convex geometry, we propose a geometrico-combinatorial description of these G -varieties which generalizes the classical examples of spherical varieties (see [Kno91]) and of normal varieties with torus action ([AH06, AHS08]), see Theorem 3.14. Moreover, for this class of G -varieties, we obtain several results on their geometry (see 3.6, 4.6, 5.1).

Context. Before stating our results, we start by recalling the definitions and the basic facts on reductive group actions. Let us fix a Borel subgroup $B \subseteq G$ and a G -variety X . Recall that the *complexity* (cf [Vin86]) of the G -action on X , denoted by $c(X)$, is defined as the transcendence degree

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over k of the field of B -invariant functions $k(X)^B$. This number does not depend on the choice of the Borel subgroup and corresponds by a result of Rosenlicht (see [Ros63], [Spr89, Satz 2.2]) to the codimension of a B -orbit of X in general position. The complexity has a remarkable property, namely, if $Z \subseteq X$ is an irreducible G -stable closed subvariety, then $c(Z) \leq c(X)$ [Tim11, Theorem 5.7].

For a G -variety X , we will say that X has a *stabilizer in general position* (or X has a (unique) general orbit) if there exist a G -stable dense open subset $X_1 \subseteq X$ and a closed subgroup $H \subseteq G$ such that for any $x \in X_1$ the isotropy group G_x is conjugated to H . In other words, this means that each G -orbit of X_1 is G -isomorphic to the homogeneous space G/H .

Moreover, a G -variety X is said to have a *trivial equivariant birational type* if there exist a variety S , a homogeneous space G/H and a G -equivariant birational map $X \dashrightarrow S \times G/H$, where G acts on the product $S \times G/H$ by the trivial action on the first factor and with the usual action on the second one. By a result of Richardson (see [Ric72]) any smooth affine G -variety has a stabilizer in general position. In particular, this applies to the case of finite dimensional rational representations of G . In addition, this is also true for G -varieties with spherical orbits. More precisely, the authors in [AB05] show that for any G -scheme \mathcal{X} of finite type over k , there exists a finite number of conjugacy classes of isotropy groups of \mathcal{X} giving rise to spherical orbits (see [AB05, Theorem 3.1]).

The classification of algebraic varieties in algebraic geometry has an equivariant analogue, namely one can distinguish two types of classification:

- (1) One determines a natural representative for each G -equivariant birational class.
- (2) Given a G -variety S , one studies (or classifies) the G -isomorphism classes of G -varieties X which are G -equivariantly birational to S .

Note that in general, we restrict ourselves to the case where the G -varieties are normal; this is the viewpoint that we will adopt thereafter. In this case, following the notation of (2), we will say that X is a G -model of S .

Several general approaches were given to study this classification problem. For the type (1), it can be reformulated in term of the relative Galois cohomology using the space of quasi-sections of a G -variety (see [PV89, Paragraph 2.5]). A description for the type (2) was obtained by Luna-Vust in the setting of embeddings of homogeneous spaces (cf [LV83]). It turns out to be effective in the case where the acting connected linear algebraic group is reductive and the complexity is ≤ 1 . A generalization for reductive group actions can be found in [Kno93b], [Tim97, Section 1]. The technics for their classifications are based on commutative algebra, especially on the theory of valuations.

Reductive group actions in small complexity still attract great interest. Let us recall that a G -variety X is said to be *spherical* if X is normal and of complexity 0 (i.e., contains an open B -orbit). The class of spherical varieties comprises several important examples appearing in algebraic geometry and representation theory. It includes the class of toric varieties (cf [Ful93]), flag varieties, horospherical varieties [PV72, Pau81, Pas08], embeddings of symmetric spaces (cf [Sat60, Vus90]), determinantal varieties, wonderful compactifications (cf [CP83, Lun96]) ... We refer to [Kno91] for a description of spherical varieties in terms of colored fans and [Tim97] for a generalization to the case of complexity one. They gave altogether a complete classification for the type (2) in the case where the complexity is ≤ 1 .

The description of a spherical variety X in [Kno91] uses the geometric structure of its open orbit and especially (equivariant) birational invariants attached to it, namely their lattice, colors and G -valuations (see Section 1.2 for a reminder). These invariants form the *colored equipment* of the spherical variety X .

In [Lun01] Luna proposed a description of the spherical homogeneous spaces and their colored equipment in terms of the root system of G and gave a complete classification when G is of type A. While Losev showed that the assignment between a G -isomorphism class of a spherical G -homogeneous space and its colored equipment is injective (see [Los09]), the general case following Luna is reformulated from a conjecture based on the classification of wonderful varieties [Lun01, Section 2]. This problem was solved recently by the efforts of several authors (see for instance [Akh83, Was96, Cup, BP14, BP15, BP16]) and allowed to finish the classification of type (1) for spherical varieties. Note that there exist alternative approaches for this problem, see for instance [Avd11, Avd16] for a classification in the special case of spherical subgroups contained in a Borel subgroup.

In 2006, Altmann and Hausen have developed a new theory for describing torus actions on normal affine varieties in the setting of arbitrary complexity [AH06]. Their idea involves the geometry of line bundles on a normal variety Γ (this later playing the role of a certain quotient for the torus action) and

the combinatorics coming from toric geometry. This description specializes to the known cases when the acting torus \mathbb{T} is of dimension one (see [Dol75, Pin77, Dem88, FZ03]) and intersects with the one for complexity-one reductive actions (see [KKMS73, Tim97, Tim08, Lan15]).

It is well known (see for instance [FZ03, Section 2]) that one can construct a normal surface X with a \mathbb{G}_m -action by considering the affine cone of a smooth projective algebraic curve Γ (modulo the action of a finite group). In this case, the algebra of regular functions $k[X]$ is described by a \mathbb{Q} -divisor D having positive degree on Γ via the equality

$$k[X] = \bigoplus_{m \geq 0} H^0(\Gamma, \mathcal{O}_\Gamma(\lfloor mD \rfloor)).$$

For instance (see [Dem88, Example 3.6]), if D is the divisor $\frac{1}{2}[0] - \frac{1}{3}[1] - \frac{1}{7}[\infty]$ over \mathbb{P}^1 , then we recover the hypersurface $x^2 + y^3 + z^7 = 0$ in \mathbb{A}^3 .

The formalism of *polyhedral divisors* introduced in [AH06] is a generalization of this phenomenon for the multigraded case where we consider instead of a \mathbb{Q} -divisor a piecewise linear map

$$\sigma^\vee \rightarrow \text{CaDiv}_\mathbb{Q}(\Gamma), \quad m \mapsto \mathcal{D}(m) = \sum_{Y \subseteq \Gamma} \min_{v \in \mathcal{D}_Y} \langle m, v \rangle \cdot Y$$

for describing the algebra of functions of an affine normal \mathbb{T} -variety. The set σ^\vee is a polyhedral cone living in the vector space $M_\mathbb{Q}$ generating by the character lattice of the torus \mathbb{T} and $\text{CaDiv}_\mathbb{Q}(\Gamma)$ is the vector space of Cartier \mathbb{Q} -divisors on the normal variety Γ . The polyhedron $\mathcal{D}_Y \subseteq N_\mathbb{Q} = \text{Hom}(M_\mathbb{Q}, \mathbb{Q})$ is referred as the coefficient at the prime divisor $Y \subseteq \Gamma$ of the polyhedral divisor \mathcal{D} , see the reminder in Section 2.1.

The generalization to the setting of normal \mathbb{T} -varieties presented in [AHS08] consists to consider a certain finite family \mathcal{E} of polyhedral divisors $\{\mathcal{D}^i, i \in I\}$, called *divisorial fan*, and defined on a common normal variety Γ (see [AHS08, Definition 5.2] for a precise definition). Here the coefficient family $\{\mathcal{D}_Y^i, i \in I\}$ forms a polyhedral subdivision. In particular, this notion collapses to the notion of the defining fan of a toric variety when the complexity of the torus action is 0.

Main results. In this article, we generalize the combinatorial description in [AH06, AHS08] for torus actions and the one of spherical varieties coming from the Luna-Vust theory [Kno91] to the setting of normal G -varieties with spherical orbits. We first collect some results about the equivariant birational type of such varieties. One knows for instance by [CKPR11, Theorem 2.13] that a G -variety having a stabilizer in general position has a trivial equivariant birational type after making an étale base change on a G -stable dense open subset (see also the reminder in 3.5).

A morphism $\pi : Z_1 \rightarrow Z_2$ between two varieties is called a *Galois covering* if it is dominant finite and the field extension $k(Z_1)/\pi^*k(Z_2)$ is Galois. By combining the result of Alexeev and Brion for the existence of a stabilizer in general position (see [AB05, Theorem 3.1]) and using tools from the Luna-Vust theory, we refine [CKPR11, Theorem 2.13] into a global result for normal G -varieties with spherical orbits. It can be stated as follows (see Theorem 3.6).

Theorem 0.1. *Let X be a normal G -variety with spherical orbits. Then there exist a normal G -variety \tilde{X} with spherical orbits having a trivial equivariant birational type and a G -equivariant Galois covering $\tilde{X} \rightarrow X$.*

In other words, this means that \tilde{X} admits a generically free G -equivariant action of a finite group F , the quotient \tilde{X}/F exists and is identified with X . Theorem 0.1 was originally shown by Arzhantsev (see [Arz97, Section 3, Proposition 3]) for certain affine G -varieties of complexity one with spherical orbits. Theorem 0.1 also gives a concrete picture for the classification of type (1) of normal G -varieties with spherical orbits. Indeed, it reduces the classification to the determination of birational models of the rational quotient of the total space \tilde{X} , the description of the general spherical orbit in terms of the Luna theory (cf. [Lun01]), and the description of a certain G -equivariant finite group action which is completely determined by a Galois cohomology class (see Corollary 3.7 for more details). The global version presented here allows us to deal first with the trivial equivariant birational case and then to go back to the general case via the finite group action.

We now explain how to describe combinatorially a normal G -variety with spherical orbits. Our main motivation is provided by the case of a toric variety V defined by a fan \mathcal{E}_V . In this setting $G = B = \mathbb{T}$ is

an algebraic torus and elements of \mathcal{E}_V are strongly convex polyhedral cones $\sigma \subseteq N_{\mathbb{Q}}$ living in the vector space generated by the lattice N of one-parameter subgroups of \mathbb{T} . Each element of $\sigma \in \mathcal{E}_V$ determines a \mathbb{T} -stable dense open affine subset $V_{\sigma} \subseteq V$, and vice-versa. Moreover, one can recover σ as the set of the discrete (geometric) valuations $v : k(V)^{\ast} \rightarrow \mathbb{Q}$ centered in the generic point of a \mathbb{T} -stable irreducible closed subvariety of V_{σ} . From this viewpoint, we have the equality $k[V_{\sigma}] = k[\mathbb{T}] \cap \bigcap_{v \in \sigma} \mathcal{O}_v$, where \mathcal{O}_v is the valuation ring associated with v . The Luna-Vust theory aims to exploit this observation in the general setting of reductive group actions. Note that the analogous notion for the V_{σ} 's in the context of an arbitrary G -variety is the notion of *simplicity*. Let us recall that a B -chart of a G -variety is a B -stable dense affine open subset. The G -variety is said to be *simple* if it has a B -chart intersecting any G -orbit. According to a result of Sumihiro (see [Sum74, Theorem 1], [Kno91, Theorem 1.3]), any normal G -variety is covered by G -translates of B -charts, or equivalently, admits a finite open covering of simple G -varieties.

Let X be a normal G -variety with spherical orbits having a trivial equivariant birational type and consider a B -chart X_0 . Our next result (see Theorem 0.2 below) is a description of X_0 in terms of a pair $(\mathcal{D}, \mathcal{F})$ called a *colored polyhedral divisor* (see Definition 2.4). The symbol \mathcal{D} denotes a polyhedral divisor defined on a certain birational model Γ of the rational quotient of X by G . More precisely, the polyhedral divisor \mathcal{D} describes the algebra of U -invariants $k[X_0]^U$ graded by the B -eigencharacters of $k(X)$, where U is the unipotent radical of the Borel subgroup B . The finite set \mathcal{F} consists of *colors* (i.e., prime B -divisors of X that are not G -stable) that intersect the open subset X_0 .

Similarly to the toric case, one can define the k -algebra $k[X_0]$ by considering valuations of $k(X)$ that are centered in the generic point of a G -stable closed irreducible subvariety or in color of X , namely that $k[X_0]$ is equal to a ring intersection

$$(k(\Gamma) \otimes_k k[\Omega_0]) \cap \bigcap_{D \in \mathcal{D}} \mathcal{O}_{v_D} \cap \bigcap_{v \in C(\mathcal{D}) \cap \mathcal{Q}_{\Sigma}} \mathcal{O}_v \subseteq k(X)$$

depending on the combinatorial datum $(\mathcal{D}, \mathcal{F})$. Here Ω_0 stands for the open B -orbit of the general orbit of X and \mathcal{Q}_{Σ} denotes the set of G -valuations of $k(X)$. We refer to Section 2.1 for further details.

Moreover, we need to know how to characterize combinatorially the B -equivariant open immersion of B -charts. For instance, in the toric situation, the choice of a face τ of the polyhedral cone $\sigma \in \mathcal{E}_V$ gives a natural \mathbb{T} -equivariant open immersion $V_{\tau} \rightarrow V_{\sigma}$. In particular, if σ_1, σ_2 are two elements of \mathcal{E}_V , then the intersection $\sigma_1 \cap \sigma_2$ is a mutual face of σ_1 and σ_2 , and therefore V is constructed by gluing the V_{σ} 's via the natural maps $V_{\sigma_1} \leftarrow V_{\sigma_1 \cap \sigma_2} \rightarrow V_{\sigma_2}$. A keypoint is that all of these open immersions are given by the localization of a \mathbb{T} -eigenfunction. We will say that a B -chart X_0 of the G -variety X is *good* if any color of X intersecting X_0 contains a G -orbit. By combining Theorem 2.8 and Lemma 2.15, we obtain in our setting the following result.

Theorem 0.2. *Any B -chart of a normal G -variety X with spherical orbits having a trivial equivariant birational type arises from a colored polyhedral divisor and vice-versa. Moreover, if $X_0 \subseteq X'_0$ are two B -charts of X and if X_0 is good, then X_0 is obtained from X'_0 by the localization of finitely many B -eigenfunctions.*

In the general situation (i.e. X has a non-trivial equivariant birational type), we have a G -equivariant Galois covering $\gamma : \tilde{X} \rightarrow X$ (see Theorem 0.1). Since the general orbit of \tilde{X} is spherical and therefore its G -equivariant automorphism group is a diagonalizable group (see [BP87, Section 5.2], [Kno91, Theorem 6.1]), the Galois group F of γ can be chosen to be abelian finite. Also, we may assume that γ is trivial if and only if its Galois cohomology class is trivial. In this case, γ is referred as a *splitting* of X .

The geometrico-combinatorial approach that we present in this article is to consider a finite set of colored polyhedral divisors \mathcal{E} that we will call a *colored divisorial fan*. This set encodes the geometry of a G -stable open covering by simple G -varieties of \tilde{X} and the F -action on \tilde{X} (given by the splitting γ) in order to determine X as the quotient \tilde{X}/F .

In the case where G is a torus \mathbb{T} , the splitting γ is the identity map and \mathcal{E} corresponds exactly to the divisorial fan introduced in [AHS08] for describing normal \mathbb{T} -varieties. Note that we have exactly the same face relations (compare [AHS08, Definition 5.1] and Remark 2.23) except that in the general case where G is possibly not a torus they are allowed when the corresponding B -charts are good. Indeed, the face relations between two colored polyhedral divisors are the result of passing from one to the other

corresponding B -chart by a finite number of successive localizations of a B -eigenfunction and depend therefore crucially on Theorem 0.2.

In addition, the intersection of two good B -charts is not a good B -chart in general. However, one can always transform a B -chart of \tilde{X} into a good B -chart by removing colors that do not contain a G -orbit (see Theorem 2.8). In Proposition 2.21 we translate it combinatorially and define an *operator of goodification* $(\mathcal{D}, \mathcal{F}) \mapsto (\mathcal{D}, \mathcal{F})^g$ in a such way that $(\mathcal{D}, \mathcal{F})^g$ describes the resulting good B -chart obtained from that of $(\mathcal{D}, \mathcal{F})$. Thus, in analogy with the toric case, the colored divisorial fan \mathcal{E} will consist of a finite set of colored polyhedral divisors on a common smooth projective variety Γ stable by goodification, by intersection and such that for all $(\mathcal{D}, \mathcal{F}), (\mathcal{D}', \mathcal{F}') \in \mathcal{E}$ the corresponding natural maps between B -charts

$$X_0(\mathcal{D}, \mathcal{F}) \leftarrow X_0(\mathcal{D} \cap \mathcal{D}', \mathcal{F} \cap \mathcal{F}')^g \rightarrow X_0(\mathcal{D}', \mathcal{F}')$$

are B -equivariant open immersions (see Definition 2.22 for more information).

Moreover, we ask that \mathcal{E} satisfies the additional condition (see Condition (iv) of loc. cit.) corresponding to the separateness and that any element is F -stable for the natural F -action on the valuation set (see Definition 3.12). To mention this latter property, we will say that \mathcal{E} is a colored divisorial fan defined on the triple $(\Gamma, \mathcal{S}, \gamma)$, where here \mathcal{S} denotes the Luna invariant attached to the general G -orbit Ω . Considering γ as a rational map, it can be defined as a generically G -equivariant F -action on the space $\Gamma \times \Omega$. Thus the datum γ encodes a priori a G -equivariant birational information. Our classification result yielding a 'toric picture' of the classification of type (2) for normal G -varieties with spherical orbits can be stated as follows (see Theorem 3.14).

Theorem 0.3. *Colored divisorial fans are the geometrico-combinatorial realizations of normal G -varieties with spherical orbits. For any normal G -variety X with spherical orbits there exist a splitting $\gamma: \tilde{X} \rightarrow X$ and a colored divisorial fan $\mathcal{E} = \mathcal{E}_X$ on a triple $(\Gamma, \mathcal{S}, \gamma)$ attached to it. The $(G \times F)$ -variety \tilde{X} is $(G \times F)$ -birational to $\Gamma \times \Omega$ and is covered by the G -translates of B -charts $X_0(\mathcal{D}, \mathcal{F})$ for any colored polyhedral divisor $(\mathcal{D}, \mathcal{F})$ running through \mathcal{E} . Moreover, if we let*

$$X(\mathcal{D}, \mathcal{F}) := G \cdot X_0(\mathcal{D}, \mathcal{F}), \text{ then we have } X(\mathcal{D}, \mathcal{F}) \cap X(\mathcal{D}', \mathcal{F}') = X(\mathcal{D} \cap \mathcal{D}', \mathcal{F} \cap \mathcal{F}') \text{ in } \tilde{X}$$

for all $(\mathcal{D}, \mathcal{F}), (\mathcal{D}', \mathcal{F}') \in \mathcal{E}$. Each G -stable open subvariety $X(\mathcal{D}, \mathcal{F})$ is F -stable.

Conversely, every colored divisorial fan defines a normal G -variety with spherical orbits.

By extending the construction of Knop for simple spherical varieties (see the proof of [Kno91, Theorem 3.1]) to our setting (see Theorem 2.27), one can define explicitly each simple G -variety $X(\mathcal{D}, \mathcal{F})$ as a locally closed G -stable subvariety in the projectivization $\mathbb{P}(V)$ of a finite dimensional rational G -module V by choosing generators of the multigraded algebra associated with the polyhedral divisor \mathcal{D} . Hence the abstract G -variety \tilde{X} associated with \mathcal{E} can be thought as the gluing of those G -varieties $X(\mathcal{D}, \mathcal{F})$ so that their intersections agree with the intersections of colored polyhedral divisors. Moreover, in Theorem 3.14 we characterize the completeness property for normal G -varieties with spherical orbits in terms of the colored divisorial fan in the same way as in [AHS08, Section 7].

Following the philosophy in [AHS08], applications of our construction are expected where the combinatorics of toric varieties have proved their usefulness. We refer the reader to [AIPSV12, Per14] for surveys on the geometry of spherical varieties and \mathbb{T} -varieties. Especially in Theorem 4.6, we describe explicitly the class group of a normal G -variety X with spherical orbits in terms of its colored divisorial fan (see [FZ03, Theorem 4.22], [PS11, Corollary 3.15], [LT16, Corollary 2.12] for former cases). Using the Riemann-Hurwitz formula for finite covering of algebraic varieties, we give an explicit description of the Weil divisor $\sharp F \cdot K_X$ (see Theorem 5.1), where K_X is a canonical divisor of X , in terms of the ramification indices of the splitting. In particular, this formula is a first step toward the classification of Fano varieties in this setting as it was studied in some particular cases in [Bat94, Pas08, Pas10, Sue14, GH]. We believe that the combinatorial description developed in this article can be useful to describe the deformation theory of spherical varieties as in [AB05, AB06]. We also refer to [Arz97, Arz02, Arz02b] for other results on the geometry of normal G -varieties with spherical orbits.

Content of the article. Let us give a brief summary of the contents of each section. In Section 1, we introduce the notation for reductive group actions by following the viewpoint of the Luna-Vust theory. Section 2 is devoted to establish Theorems 0.1 and 0.3 for the case where the G -equivariant birational

type is trivial. We also introduce some combinatoric tools as the concept of colored polyhedral divisor. In Section 3, we investigate the equivariant birational classification of a normal G -variety with spherical orbits and in particular we prove Theorem 0.2 and 0.3. Finally, Sections 4 and 5 provide some applications. After translating the local structure theorem into the language of colored polyhedral divisors (see Theorem 4.2), we determine the class group of a normal G -variety with spherical orbits in 4.6 and obtain information on the canonical class in the last section.

Notation. By a *variety* (resp. a *linear algebraic group*) we mean an integral separated scheme (resp. a reduced affine group scheme) of finite type over k . All subgroups of a linear algebraic group are assumed to be closed. Given a variety X , we denote by $k[X] = \Gamma(X, \mathcal{O}_X)$ its algebra of regular global functions and by $k(X)$ its field of rational functions.

The letter G stands for a connected reductive linear algebraic group. Changing G by a finite covering, we may restrict to the case where G is *simply-connected*. This latter condition means that G is a direct product $C \times G^{ss}$, where C is an algebraic torus and G^{ss} is a simply-connected semi-simple linear algebraic group. We will consider a maximal torus T in a Borel subgroup B of G . We denote by U the unipotent radical of B so that $B = TU$ and by Δ the set of simple roots with the respect to the pair (B, T) .

We will use [Tim11] as reference for the geometry of homogeneous spaces.

1. PRELIMINARIES

In this section, we recall some basic notions on reductive group actions that we will use in this paper.

1.1. Valuations and colors. Many constructions that we will encounter deal with valuations and colors. From the viewpoint of the Luna-Vust theory [Tim11, Chapter 12], they constitute the basic material for describing the varieties endowed with an action of the connected reductive group G .

Let X be an integral G -scheme of finite type over k and let $L \subseteq G$ be a subgroup. A *discrete valuation* of $k(X)$ is a group homomorphism

$$v : (k(X)^*, \times) \rightarrow (\mathbb{Q}, +)$$

with kernel containing the subgroup k^* and image $a\mathbb{Z}$ for some $a \in \mathbb{Q}$, such that $v(f_1 + f_2)$ is equal to $\min\{v(f_1), v(f_2)\}$ for all $f_1, f_2 \in k(X)^*$ satisfying $f_1 + f_2 \neq 0$. The valuation v of $k(X)$ has a *center* in X if there exists a schematic point $\xi \in X$ such that the valuation ring \mathcal{O}_v dominates \mathcal{O}_ξ , i.e., $\mathcal{O}_\xi \subseteq \mathcal{O}_v$ and $\mathfrak{m}_\xi \subseteq \mathfrak{m}_v$ for the corresponding maximal ideals.

Assume that X is a normal variety. Then every prime divisor D on the variety X determines a discrete valuation on $k(X)$ called the *vanishing order* along D . We denote it by v_D . Note that the generic point of D is a center of v_D . The *geometric valuations* are those of the forms αv_D for any possible choice of normal varieties X' such that $k(X) = k(X')$, prime divisors $D \subseteq X'$ and scalars $\alpha \in \mathbb{Q}_{>0}$. The valuation v is said to be *L -invariant* (or simply called an *L -valuation*) if it is geometric and if further the equalities $v(g \cdot f) = v(f)$ hold for all $f \in k(X)^*$ and $g \in L$. It is called *central* if its restriction to the subfield of invariants $k(X)^B$ is trivial.

Let us introduce special subvarieties of the G -variety X which will play an important role later on. An *L -cycle* (or *L -germ*) of X is an L -stable irreducible closed subvariety of X , an *L -divisor* is an L -cycle of codimension one, and a *color* is a B -divisor which is not G -stable. In particular, every L -divisor on X defines an L -valuation on $k(X)$.

We now introduce the scheme of geometric localities. It will be useful for theoretical issues. We refer to [LV83, Section 1] for more information.

Let \mathcal{X} be a variety. A (geometric) *locality* of $k(\mathcal{X})$ is a local ring associated with a prime ideal of a finitely generated normal subalgebra $A \subseteq k(\mathcal{X})$ having field of fractions $k(\mathcal{X})$. The set of localities $\text{Mod}(\mathcal{X})$ of $k(\mathcal{X})$ is naturally endowed with a structure of scheme over k where the possible affine schemes $\text{Spec } A$ are considered as open subsets. Note that the scheme $\text{Mod}(\mathcal{X})$ is in general not separated over k .

A (G -)model of \mathcal{X} is a normal (G -)variety equipped with a (G -equivariant) birational map

$$X \dashrightarrow \mathcal{X},$$

yielding an identification of (G) -algebras over k between $k(X)$ and $k(\mathcal{X})$. We denote by $\text{Mod}_G(\mathcal{X})$ the G -scheme of geometric localities of \mathcal{X} (also called the universal G -model). As a set, $\text{Mod}_G(\mathcal{X})$ consists of localities $\mathcal{O}_{X,Y} \subseteq k(\mathcal{X})$, where Y is a prime cycle of a G -model X of \mathcal{X} . It corresponds to the maximal open G -stable subset of $\text{Mod}(\mathcal{X})$ in which the G -action on $\text{Mod}(\mathcal{X})$ is regular. As a summary, any (G) -model of \mathcal{X} can be thought as a (G) -stable separated open subschemes of finite type over k of $\text{Mod}_G(\mathcal{X})$, and vice-versa.

Local information of the action of the connected reductive group G on the normal variety X can be obtained from their B -charts, that is, their affine B -stable dense open subsets. We will consider B -charts of the G -scheme $\text{Mod}_G(\mathcal{X})$ which will be simply B -charts of the G -models of \mathcal{X} . The G -variety X will be called *simple* if it possesses a B -chart intersecting every G -orbit. This terminology makes sense, since according to the Sumihiro theorem (see [Sum74, Theorem 1], [Kno91, Theorem 1.3]), every normal G -variety is a finite open union of simple G -varieties.

1.2. Spherical subgroups. In [Lun01] Luna attached to any spherical subgroup of G a discrete invariant depending on the root system of G : the prominent combinatorial object occuring is a homogeneous spherical datum. It was shown that these objects classify the conjugacy classes of spherical subgroups of G [Los09, BP16]. The aim of this subsection is to give a brief overview of this description. We recall that a subset C of a finite dimensional \mathbb{Q} -vector space E is a *polyhedral cone* if there exist vectors $v_1, \dots, v_d \in E$ such that $C = \mathbb{Q}_{\geq 0}v_1 + \dots + \mathbb{Q}_{\geq 0}v_d$.

Let $\Omega = G/H$ be a spherical G -homogeneous space. We define the lattice M as the set of B -eigencharacters of the B -algebra $k(\Omega)$. Denote by $N = \text{Hom}(M, \mathbb{Z})$ the dual lattice and by $M_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} M$, $N_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} N$ the associated dual vector spaces. The set of B -divisors of Ω consists of the irreducible components of the complement of the open B -orbit in Ω . It forms the set of colors \mathcal{F} of Ω .

Any valuation v on $k(\Omega)$ and any B -eigenfunction $f \in k(\Omega)$ gives a pairing $\langle \varrho(v), \chi_f \rangle = v(f)$, where χ_f is the B -weight associated with f . This expression is well-defined since f is uniquely determined by χ_f up to the multiplication of a nonzero constant.

It is known by [LV83, Proposition 7.4] that the map ϱ is injective on the set of G -valuations \mathcal{V} of $k(\Omega)$. We will again denote by \mathcal{V} the image $\varrho(\mathcal{V})$. The subset \mathcal{V} is a full dimensional cosimplicial polyhedral cone in $N_{\mathbb{Q}}$ and admits therefore a presentation

$$\mathcal{V} = \bigcap_{\gamma \in \Sigma} \{v \in N_{\mathbb{Q}} \mid \langle \gamma, v \rangle \geq 0\},$$

where Σ is a finite set of linear independent primitive lattice vectors of M (compare with [BP87]). The set Σ is called the *set of spherical roots* of Ω . By [Los09, Theorem 1], the triple (M, Σ, \mathcal{F}) determines uniquely the spherical homogeneous space Ω up to a G -isomorphism.

Let $\alpha \in \Delta$ be a simple root and $\mathcal{F}(\alpha) = \mathcal{F}_{G/H}(\alpha)$ be the set of colors $D \subseteq G/H$ such that the corresponding minimal parabolic subgroup P_{α} moves D , i.e., $P_{\alpha} \cdot D \neq D$. The parabolic subgroup associated with the subset

$$\Delta^p = \{\alpha \in \Delta \mid \mathcal{F}(\alpha) = \emptyset\}$$

is the subgroup of G preserving the open B -orbit of Ω . Moreover, denoting by D_1, \dots, D_s the distinct colors such that

$$\{\alpha \in \Delta \mid P_{\alpha} \cdot D_i \neq D_i\} \cap \Sigma \neq \emptyset$$

for any i , we define the family \mathbf{A} as $(\varrho(D_i))_{1 \leq i \leq s}$.

Summarizing, to every spherical G -homogeneous space $\Omega = G/H$ we may attach

$$\mathcal{S}_{\Omega} = (\Delta_{\Omega}^p, \Sigma_{\Omega}, \mathbf{A}_{\Omega}, M_{\Omega}) = (\Delta^p, \Sigma, \mathbf{A}, M).$$

It was shown that any datum \mathcal{S}_{Ω} satisfies the combinatorial conditions of a *homogeneous spherical datum*; we refer to [Lun01, §2] for the list of axioms. Note that from \mathcal{S}_{Ω} one can recover the set of colors of Ω (see [Lun01, §2.3]). In addition, we have the following important result of classification, see [Lun01, Los09, BP16].

Theorem 1.1. *The map*

$$\Omega \mapsto \mathcal{S}_\Omega = (\Delta_\Omega^p, \Sigma_\Omega, \mathbf{A}_\Omega, M_\Omega)$$

from the class of spherical G -homogeneous spaces to the set of homogeneous spherical data is well-defined. It induces an injective map on the set of G -isomorphism classes of spherical G -homogeneous spaces. Every homogeneous spherical datum of G is geometrically realizable by a spherical G -homogeneous space via the map $\Omega \mapsto \mathcal{S}_\Omega$.

The next two examples deal with reductive spherical subgroups of SL_2 .

Example 1.2. We consider the natural diagonal action of SL_2 on $\mathbb{P}^1 \times \mathbb{P}^1$. Let T be the subgroup of diagonal matrices and B be the subgroup of upper triangular matrices. Then the spherical homogeneous space SL_2/T is identified with

$$\mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{diag}(\mathbb{P}^1) = \{([x_0 : 1], [y_0 : 1]) \mid x_0 \neq y_0\}.$$

We have $M = \mathbb{Z}\alpha$, where α is the character $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto a$ and $k[\chi^{-\alpha}, \chi^\alpha]$ is the subalgebra of $k(\mathrm{SL}_2/T)$ generated by $k(\mathrm{SL}_2/T)^{(B)}$, where $\chi^\alpha = (x_0 - y_0)^{-1}$. Moreover, SL_2/T has two colors by intersecting with the two subsets $D_+ = \mathbb{P}^1 \times \{[0 : 1]\}$ and $D_- = \{[0 : 1]\} \times \mathbb{P}^1$. Note that $\langle \varrho(D_\pm), \alpha \rangle = 1$. Finally, the homogeneous spherical datum $\mathcal{S} = (\Delta^p, \Sigma, \mathbf{A}, M)$ of SL_2/T is given by $\Delta^p = \emptyset$, $\Sigma = \{\alpha\}$, $\mathbf{A} = \{D_-, D_+\}$, and $M = \mathbb{Z}\alpha$.

Example 1.3. We consider the projective plane $\mathbb{P}^2 \simeq \mathbb{P}(S^2V)$, where V is a two-dimensional vector space over k . Here S^2V is the space of binary forms $S^2V = k v^2 \oplus k v w \oplus k w^2$. The map

$$\tau : \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathrm{diag}(\mathbb{P}^1) \rightarrow \mathbb{P}^2 \setminus E, ([x_0 : 1], [y_0 : 1]) \mapsto [(x_0 v + w)(y_0 v + w)]$$

is identified with the natural projection $\mathrm{SL}_2/T \rightarrow \mathrm{SL}_2/H$, where H is the normalizer of T in SL_2 . The subset E equal to $\{[(x_0 v + y_0 w)^2] \mid [x_0 : y_0] \in \mathbb{P}^1\}$ is the space of degenerate binary forms. The morphism τ is a covering involution which sends the union $D_- \cup D_+$ of the colors of SL_2/T onto the unique color D of SL_2/H . Finally, the homogeneous spherical datum $\mathcal{S} = (\Delta^p, \Sigma, \mathbf{A}, M)$ of SL_2/H is given by $\Delta^p = \emptyset$, $\Sigma = \{2\alpha\}$, $\mathbf{A} = \emptyset$, and $M = 2\mathbb{Z}\alpha$.

Example 1.4. *Horospherical homogeneous spaces.* A closed subgroup $H \subseteq G$ is said to be *horospherical* if H contains a maximal unipotent subgroup of G . Then in this case, the normalizer $P = N_G(H)$ is a parabolic subgroup corresponding to a set of simple roots Δ^p and P/H is an algebraic torus \mathbb{T} with character lattice M . Note that G/H is spherical and is equal to the parabolic induction $G \times^P \mathbb{T}$. Hence the homogeneous spherical datum \mathcal{S} of H is $(\Delta^p, \emptyset, \emptyset, M)$ (see [Pas08, Proposition 2.4]).

2. COMBINATORICS

Let \mathcal{X} be the G -variety $S \times \Omega$. The action on \mathcal{X} is defined by acting G trivially on the variety S and by left translations on the spherical homogeneous space $\Omega = G/H$. We will denote by $\mathcal{S} = (\Delta^p, \Sigma, \mathbf{A}, M)$ the homogeneous spherical datum corresponding to Ω .

In this section, we classify the normal G -varieties for the case of G -models of \mathcal{X} by using the language of polyhedral divisors (see [AH06]). The main idea of this description is based on the fact that we may express the algebra of a B -chart of a given G -model of \mathcal{X} as an intersection of discrete valuations rings.

2.1. Colored polyhedral divisors. Let us introduce the geometric environment where the G -valuations of $k(\mathcal{X})$ are represented [Tim11, Section 20.1]. We denote by \mathcal{Q} the *hyperspace* associated with the pair (S, N) . It is defined as the quotient of $\varsigma(S) \times N_{\mathbb{Q}} \times \mathbb{Q}_{\geq 0}$ by the equivalence relation \sim given by

$$(s, p, \ell) \sim (s', p', \ell') \text{ if and only if } s = s', p = p', \ell = \ell' \text{ or } p = p', \ell = \ell' = 0,$$

where $\varsigma(S)$ is the set of geometric valuations of $k(S)$ considered up to proportionality. The equivalence class of (s, p, ℓ) will be denoted by $[s, p, \ell]$. Note that the element $[s, p, 0]$ does not depend on s and we will write it by the symbol $[\cdot, p, 0]$. Let us consider the short exact sequence of abelian groups

$$0 \rightarrow k(S)^* \rightarrow k(\mathcal{X})^{(B)} \rightarrow M \rightarrow 0,$$

where $k(\mathcal{X})^{(B)}$ is the multiplicative group of the rational B -eigenfunctions on \mathcal{X} . The arrow $k(\mathcal{X})^{(B)} \rightarrow M$ is the map sending a B -eigenfunction to its B -weight. Let $A_M \subseteq k(\mathcal{X})$ be the subalgebra generated by $k(\mathcal{X})^{(B)}$. Then choosing a lifting $M \rightarrow k(\mathcal{X})^{(B)} \cap k(\Omega)$, $m \mapsto \chi^m$ we obtain the decomposition

$$A_M = \bigoplus_{m \in M} k(S) \otimes \chi^m.$$

Every G -valuation v of $k(\mathcal{X})$ is uniquely determined by its values on A_M (see [Tim11, Corollary 19.13]) and therefore [Tim11, Proposition 20.7] defines an element $[s, a, b] \in \mathcal{Q}$ via $v(f \otimes \chi^m) = b \cdot s(f) + \langle m, a \rangle$, where $f \in k(S)^*$ and $m \in M$. We denote by \mathcal{Q}_Σ the set of G -valuations identified with a subset of \mathcal{Q} .

The next proposition determines \mathcal{Q}_Σ in terms of the homogeneous spherical datum \mathcal{S} . This result seems to be well known for experts. By convenience we include here a short proof.

Proposition 2.1. *Let \mathcal{V} be the valuation cone of the spherical homogeneous space Ω (which is completely determined by the set of spherical roots Σ). Then we have the equality*

$$\mathcal{Q}_\Sigma = \{[s, a, b] \in \mathcal{Q} \mid s \in \varsigma(S), (a, b) \in \mathcal{V} \times \mathbb{Q}_{\geq 0}\}.$$

Proof. Let $v = [s, a, b] \in \mathcal{Q}_\Sigma$. The restriction of v to the subfield $k(\Omega) \subseteq k(\mathcal{X})$ is a G -valuation. This implies that $a \in \mathcal{V}$. Conversely, let us show that

$$\{[s, a, b] \in \mathcal{Q} \mid s \in \varsigma(S), (a, b) \in \mathcal{V} \times \mathbb{Q}_{\geq 0}\} \subseteq \mathcal{Q}_\Sigma.$$

We will define two valuations v_1 and v_2 on the function field $k(\mathcal{X})$ depending on an $a \in \mathcal{V}$ as follows. We first consider an embedding X_a of Ω with the property that the complement of the open orbit is a G -orbit D_a of codimension one with vanishing order equal to a . Such embedding always exists (see [LV83, 3.3, 7.5, 8.10]). Let Γ be a model of S and denote by $Y \subseteq \Gamma$ a prime divisor. Then v_1 and v_2 are respectively the vanishing orders of $\Gamma \times D_a$ and $Y \times X_a$ in the G -model $\Gamma \times X_a$ of \mathcal{X} . This implies that $[s, a, 0], [s, 0, 1] \in \mathcal{Q}_\Sigma$ where $a \in \mathcal{V}$ and $s \in \varsigma(S)$. We conclude by using [Tim11, Theorem 20.3]. \square

In the next paragraphs, we will introduce the notion of colored polyhedral divisors (compare with [AH06, Section 2]). This latter will appear as the combinatorial counterpart of a B -chart.

Let $\sigma \subseteq N_\mathbb{Q}$ be a strongly convex polyhedral cone (that is a polyhedral cone which contains no line) and let Γ be a model of S . Recall that a *polytope* of $N_\mathbb{Q}$ is the convex hull of a non-empty finite subset of $N_\mathbb{Q}$. A σ -polyhedral divisor on Γ is a formal sum

$$\mathcal{D} = \sum_{Y \subseteq \Gamma} \mathcal{D}_Y \cdot Y,$$

where \mathcal{D}_Y is empty or a σ -polyhedron (i.e., $\mathcal{D}_Y \subseteq N_\mathbb{Q}$ is a Minkowski sum of σ and a polytope), $Y \subseteq \Gamma$ runs through the set of prime divisors of Γ , and $\mathcal{D}_Y = \sigma$ for all but finitely many prime divisors $Y \subseteq \Gamma$. The complement in Γ of the union of prime divisors $Y \subseteq \Gamma$ such that $\mathcal{D}_Y = \emptyset$ and the cone σ are called respectively the *locus* and the *tail* of \mathcal{D} ; we will denote them by $\text{Loc}(\mathcal{D})$ and $\text{Tail}(\mathcal{D})$ if the notation is not explicitly specified. Let us denote by σ^\vee the dual cone defined as the subset

$$\sigma^\vee = \{m \in M_\mathbb{Q} \mid \forall v \in \sigma, \langle m, v \rangle \geq 0\}$$

which consists of the linear forms of $M_\mathbb{Q}$ that are non-negative on σ . The *evaluation* at the vector $m \in \sigma^\vee$ is the \mathbb{Q} -divisor

$$\mathcal{D}(m) = \sum_{Y \subseteq \Gamma} \min_{v \in \mathcal{D}_Y} \langle m, v \rangle \cdot Y.$$

The expression

$$\mathcal{A} = \bigoplus_{m \in \sigma^\vee \cap M} \mathcal{O}_{\text{Loc}(\mathcal{D})}(\mathcal{D}(m)) \otimes \chi^m$$

naturally defines an M -graded $\mathcal{O}_{\text{Loc}(\mathcal{D})}$ -algebra where the multiplication on the homogeneous elements is induced by the multiplication on $k(S)$. The algebra of global sections $A(\Gamma, \mathcal{D}) := \Gamma(\text{Loc}(\mathcal{D}), \mathcal{A})$ is the *associated algebra of \mathcal{D}* . It naturally embeds in A_M as an M -graded subalgebra.

The properness is a technical condition on the polyhedral divisor \mathcal{D} which ensures that the algebra $A(\Gamma, \mathcal{D})$ is of finite type over k and that its field of fractions is equal to that of A_M [AH06, Theorem 3.1]. Note that this condition on \mathcal{D} is needed for the finite generation condition even in the case where Γ is an algebraic curve, see the counterexample of Knop in [Tim11, Remark 16.22], [Kno93a]. It is defined as follows.

Definition 2.2. The σ -polyhedral divisor \mathcal{D} is said to be *proper* (cf. [AH06, Definition 2.7]) if it satisfies the following additional properties.

- The locus $\text{Loc}(\mathcal{D})$ is a *semiprojective variety*, i.e., it is projective over an affine variety.
- For all $m \in \sigma^\vee$, the \mathbb{Q} -divisor $\mathcal{D}(m)$ is a *semiample* Cartier \mathbb{Q} -divisor, i.e., a multiple of $\mathcal{D}(m)$ corresponds to a basepoint-free line bundle.
- For all $m \in M_{\mathbb{Q}}$ in the relative interior of σ^\vee , the \mathbb{Q} -divisor $\mathcal{D}(m)$ is *big*, i.e., a multiple of $\mathcal{D}(m)$ admits a global section with affine complement zero locus.

In particular, any polyhedral divisor with Cartier evaluations and affine locus is proper. The following result determines which multigraded algebra is described by a proper polyhedral divisor. They correspond geometrically to algebras of regular functions of normal affine varieties with an effective torus action.

Theorem 2.3. [AH06, Theorem 3.4] *Let $\sigma \subseteq N_{\mathbb{Q}}$ be a strongly convex polyhedral cone. Let A be a normal M -graded subalgebra of A_M of finite type over k with weight cone σ^\vee and field of fractions equal to that of A_M . Then there exists an open semiprojective subvariety $\Gamma \subseteq \text{Mod}(S)$ and a proper σ -polyhedral divisor \mathcal{D} on Γ such that $A = A(\Gamma, \mathcal{D})$.*

Let us introduce various objects attached to the σ -polyhedral divisor \mathcal{D} . The *Cayley cones* of \mathcal{D} are the cones $C_Y(\mathcal{D}) \subseteq N_{\mathbb{Q}} \oplus \mathbb{Q}$ generated by the union of $(\sigma, 0)$ and $(\mathcal{D}_Y, 1)$, where $Y \subseteq \Gamma$ is a prime divisor. The *hypercone* associated with \mathcal{D} is the subset

$$C(\mathcal{D}) = \{[v_Y, a, b] \in \mathcal{D} \mid Y \subseteq \Gamma \text{ prime divisor, } (a, b) \in C_Y(\mathcal{D})\}.$$

We define in an obvious way the relative interior of $C(\mathcal{D})$ as the complement of the proper ‘hyperfaces’ of $C(\mathcal{D})$. Let us define the combinatorial objects which allow us to classify the simple G -models of \mathcal{X} .

Definition 2.4. Considering the set of colors \mathcal{F}_Ω of Ω , a pair (σ, \mathcal{F}) , where \mathcal{F} is a subset of \mathcal{F}_Ω , is a *colored cone*¹ (cf. [Kno91, Section 3]) if $\varrho(\mathcal{F})$ does not contain 0 and σ is a strongly convex polyhedral cone generated by the union of $\varrho(\mathcal{F})$ and a finite subset of \mathcal{V} . Colored cones are combinatorial objects related to the classification of simple spherical varieties, see for instance [Tim11, Section 15.1].

The pair $(\mathcal{D}, \mathcal{F})$ is a *colored σ -polyhedral divisor* on Γ if the following hold.

- (σ, \mathcal{F}) is a colored cone.
- \mathcal{D} is a proper σ -polyhedral divisor.
- The hypercone $C(\mathcal{D})$ is generated by the union of $C(\mathcal{D}) \cap \mathcal{D}_\Sigma$ and $\varrho(\mathcal{F})$. In particular, the set of vertices of \mathcal{D}_Y is contained in \mathcal{V} , for any prime divisor $Y \subseteq \Gamma$.

We denote by $\text{CPDiv}(\Gamma, \mathcal{S})$ the set of colored polyhedral divisors with the respect to the normal semiprojective variety $\Gamma \subseteq \text{Mod}(S)$ and the homogeneous spherical datum \mathcal{S} .

We now consider the subsets $\mathcal{U} \subseteq \mathcal{D}_\Sigma$ and $\mathcal{F} \subseteq \mathcal{F}_\Omega$, where \mathcal{F}_Ω is seen as the set of colors of $\text{Mod}_G(\mathcal{X})$. With these two data, one can construct a B -stable subalgebra of $k(\mathcal{X})$. Denote by \mathcal{O}_v the local ring associated with the valuation v . We then let

$$R(\mathcal{U}, \mathcal{F}) = (k(\Gamma) \otimes_k k[\Omega_0]) \cap \bigcap_{D \in \mathcal{F}} \mathcal{O}_{v_D} \cap \bigcap_{v \in \mathcal{U}} \mathcal{O}_v \subseteq k(\mathcal{X}),$$

where Ω_0 is the open B -orbit of Ω . The next lemma follows from an adaptation of the results in [LV83, Section 8]. It gives conditions for the affine scheme $X_0 = \text{Spec } R(\mathcal{U}, \mathcal{F})$ to be a B -chart of $\text{Mod}_G(\mathcal{X})$.

¹To have more flexibility, we take a different viewpoint by modifying slightly the definition of colored cone in [Kno91, Section 3]. In our definition, we do not impose that the relative interior of σ intersects \mathcal{V} . The reason is that in [Kno91, Section 3] the author deals only with *minimal* B -charts of spherical varieties in order to have a perfect dictionary between colored cones and simple spherical embeddings (see [Kno91, Theorem 3.1]). We refer to [Tim11, Section 15.1 and Remark 14.3] for more information.

Lemma 2.5. [Tim11, Theorem 13.8] *Denote by E the set $\mathcal{U} \sqcup \mathcal{F}$ where \mathcal{F} is considered as a set of valuations of $k(\mathcal{X})$. The affine scheme $X_0 = \text{Spec } R(\mathcal{U}, \mathcal{F})$ is a B -chart of $\text{Mod}_G(\mathcal{X})$ if and only if the following conditions are satisfied.*

- (i) *For any finite subset $E_0 \subseteq E$, there exists a homogeneous element $\xi \in A_M$ such that for all $v \in E$ and $w \in E_0$ we have $v(\xi) \geq 0$ and $w(\xi) > 0$.*
- (ii) *The subalgebra $R(\mathcal{U}, \mathcal{F})^U = k[R(\mathcal{U}, \mathcal{F})^{(B)}]$ is of finite type over k , where $U \subseteq G$ is the unipotent radical of B .*

Moreover, any B -chart of $\text{Mod}_G(\mathcal{X})$ arises in this way.

Actually, Conditions (i), (ii) in the preceding lemma imply that the normal algebra $R(\mathcal{U}, \mathcal{F})$ has field of fractions $k(\mathcal{X})$ and is of finite type over k , respectively, so that the affine scheme X_0 is an open subset of $\text{Mod}(\mathcal{X})$. Following the argument of [LV83, Section 8], it was shown [Tim11, Theorem 13.8] that $R(\mathcal{U}, \mathcal{F})$ is stable under the natural action of the Lie algebra of G (see [Tim11, Proposition 12.3]) making X_0 a G -model of \mathcal{X} .

We will use the next two technical lemmata from the Luna-Vust theory.

Lemma 2.6. [Tim11, Lemma 19.12] *For any $f \in k(\Gamma) \otimes_k k[\Omega_0]$ and any G -valuation $v \in \mathcal{Q}_\Sigma$, there exists a B -eigenfunction $\tilde{f} \in A_M$ such that the following hold.*

- (i) $v(\tilde{f}) = v(f)$ and $w(\tilde{f}) \geq w(f)$ for all $w \in \mathcal{Q}_\Sigma$.
- (ii) $v_D(\tilde{f}) \geq v_D(f)$ for all $D \in \mathcal{F}_\Omega$.

Lemma 2.7. [Tim11, Theorem 14.2] *Let X_0 be a B -chart of $\text{Mod}_G(\mathcal{X})$ described by a pair $(\mathcal{U}, \mathcal{F})$ as in Lemma 2.5. Let $D \in \mathcal{F}$ be a color containing a G -cycle $O \subseteq \text{Mod}_G(\mathcal{X})$ intersecting X_0 and let v be a G -valuation centered in the generic point of O . Then for any B -eigenfunction in $R(\mathcal{U}, \mathcal{F})$ such that $v(f) = 0$, we have $v_D(f) = 0$.*

The next result gives a combinatorial picture for the correspondence between G -valuations and colors, and the B -charts of \mathcal{X} by employing the language of polyhedral divisors introduced in [AH06].

Theorem 2.8. *Let $(\mathcal{D}, \mathcal{F}) \in \text{CPDiv}(\Gamma, \mathcal{S})$ be a colored polyhedral divisor on an open semiprojective subvariety $\Gamma \subseteq \text{Mod}(S)$. Then the affine scheme $X_0(\mathcal{D}, \mathcal{F}) = \text{Spec } R(C(\mathcal{D}) \cap \mathcal{Q}_\Sigma, \mathcal{F})$ is a B -chart of $\text{Mod}_G(\mathcal{X})$. Any B -chart of $\text{Mod}_G(\mathcal{X})$ arises from a colored polyhedral divisor as above. Therefore any simple G -model of \mathcal{X} is of the form*

$$X(\mathcal{D}, \mathcal{F}) := G \cdot X_0(\mathcal{D}, \mathcal{F}) \subseteq \text{Mod}_G(\mathcal{X}).$$

Moreover, the B -chart $X_0(\mathcal{D}, \mathcal{F})$ can be chosen in such a way that any color of \mathcal{F} contains a G -orbit of $X(\mathcal{D}, \mathcal{F})$. The algebra of U -invariants $k[X_0(\mathcal{D}, \mathcal{F})]^U$ is identified with $A(\Gamma, \mathcal{D})$.

Definition 2.9. A B -chart X_0 of $\text{Mod}_G(\mathcal{X})$ corresponding to a colored polyhedral divisor $(\mathcal{D}, \mathcal{F}) \in \text{CPDiv}(\Gamma, \mathcal{S})$ will be called *good* if any color of \mathcal{F} (or equivalently, intersecting X_0) contains a G -orbit of $X = G \cdot X_0$. Similarly, if X_0 is not good, a color of \mathcal{F} that contain no G -orbit will be called *bad*. Note that in the setting of normal G -varieties with horospherical orbits (see [LT16] for the complexity-one case) every B -chart is good.

Proof of Theorem 2.8. Let us show that $X_0(\mathcal{D}, \mathcal{F})$ satisfies Conditions (i) and (ii) of Lemma 2.5 for any $(\mathcal{D}, \mathcal{F}) \in \text{CPDiv}(\Gamma, \mathcal{S})$. We naturally have the following equivalences

$$\begin{aligned} \xi = f \otimes \chi^m \in R(C(\mathcal{D}) \cap \mathcal{Q}_\Sigma, \mathcal{F})^U &\Leftrightarrow \forall v \in C(\mathcal{D}) \cap \mathcal{Q}_\Sigma, \forall D \in \mathcal{F}, v(\xi) \geq 0 \text{ and } v_D(\xi) \geq 0 \\ &\Leftrightarrow \forall Y \subseteq \Gamma, \forall a \in \mathcal{D}_Y, \forall p \in \sigma, v_Y(f) + \langle m, a \rangle \geq 0 \text{ and } \langle m, p \rangle \geq 0, \end{aligned}$$

for any homogenous element ξ of A_M . Hence $R(C(\mathcal{D}) \cap \mathcal{Q}_\Sigma, \mathcal{F})^U = A(\Gamma, \mathcal{D})$ and by [AH06, Theorem 3.1], Condition (ii) is verified. Let $E_0 \subseteq E$ be a finite set. Denote by E_1 and E_2 the subsets of E_0 of central and non-central valuations, respectively. Since the elements of E_2 can be represented as elements in $\text{Tail}(\mathcal{D})$, we may find $m \in M$ such that for any $v \in E_1$ we have $v(1 \otimes \chi^m) > 0$. Moreover, let $f \in k(S)^*$ such that $v(f \otimes \chi^m) > 0$ for all $v \in E_2$. As \mathcal{D} is a proper polyhedral divisor, the field of fractions of $A(\Gamma, \mathcal{D})$ is equal to that of A_M , and so there exist homogeneous elements ξ, ξ' of $A(\Gamma, \mathcal{D})$ such that $f \otimes \chi^m = \xi/\xi'$. Hence $v(\xi) \geq 0$ for all $v \in E$ and

$$w(\xi) \geq w(\xi) - w(\xi') = w(f \otimes \chi^m) > 0$$

for all $w \in E_0$, yielding Condition (ii) of Lemma 2.5. This shows that $X_0(\mathcal{D}, \mathcal{F})$ is a B -chart of $\text{Mod}_G(\mathcal{X})$.

Conversely, let X_0 be a B -chart of $\text{Mod}_G(\mathcal{X})$. Since $k[X_0]$ is a Krull ring, we may write

$$k[X_0] = \bigcap_{D \in D(\mathcal{X}) \sqcup \mathcal{F}} \mathcal{O}_{v_D} \cap \bigcap_{v \in \mathcal{U}} \mathcal{O}_v,$$

where $D(\mathcal{X})$ stands for the set of prime divisors of \mathcal{X} that are not B -stable (see [Tim97, Section 1.4], [Tim11, Section 13]). Note that this set does not depend on the choice of \mathcal{X} and

$$k(\Gamma) \otimes_k k[\Omega_0] = \bigcap_{D \in D(\mathcal{X})} \mathcal{O}_{v_D} \text{ so that } k[X_0] = R(\mathcal{U}, \mathcal{F}).$$

Moreover, the pair $(\mathcal{U}, \mathcal{F})$ satisfies Conditions (i) and (ii) of Lemma 2.5. Let C be the linear span of $(\mathcal{U}, \mathcal{F})$ in \mathcal{Q} . It is defined as the subset

$$C = \{[s, a, b] \in \mathcal{Q} \mid s(f) + b\langle m, a \rangle \geq 0 \text{ for all } f \otimes \chi^m \in R(\mathcal{U}, \mathcal{F})^{(B)}\}.$$

Let us show that $k[X_0] = R(C \cap \mathcal{Q}_\Sigma, \mathcal{F})$. Since $\mathcal{U} \subseteq C \cap \mathcal{Q}_\Sigma$ we have obviously

$$R(C \cap \mathcal{Q}_\Sigma, \mathcal{F}) \subseteq k[X_0].$$

Let $\zeta \in k[X_0]$ and take $v \in C \cap \mathcal{Q}_\Sigma$. By Lemma 2.6, there exists $\xi \in A_M^{(B)}$ such that

$$w(\xi) \geq w(\zeta) \geq 0 \text{ and } v_D(\xi) \geq v_D(\zeta) \geq 0 \text{ for all } w \in \mathcal{U}, D \in \mathcal{F}$$

and $v(\zeta) = v(\xi)$. Remarking that these latter conditions imply that

$$\xi \in R(C \cap \mathcal{Q}_\Sigma, \mathcal{F})^U = k[X_0]^U$$

we get $v(\zeta) \geq 0$. Hence $\zeta \in R(C \cap \mathcal{Q}_\Sigma, \mathcal{F})$, yielding the desired equality $k[X_0] = R(C \cap \mathcal{Q}_\Sigma, \mathcal{F})$.

In the sequel, we may assume that $\mathcal{U} = C \cap \mathcal{Q}_\Sigma$. Now using Theorem 2.3, there exist an open semiprojective subvariety $\Gamma \subseteq \text{Mod}(S)$ and a proper polyhedral divisor \mathcal{D} on Γ such that $k[X_0]^U = A(\Gamma, \mathcal{D})$.

We claim that $k[X_0] = R(C(\mathcal{D}) \cap \mathcal{Q}_\Sigma, \mathcal{F})$. Indeed, by definition of the set C , we have $C(\mathcal{D}) \cap \mathcal{Q}_\Sigma \subseteq C \cap \mathcal{Q}_\Sigma$ and thus $k[X_0] \subseteq R(C(\mathcal{D}) \cap \mathcal{Q}_\Sigma, \mathcal{F})$. Let us assume that there exists $\zeta \in R(C(\mathcal{D}) \cap \mathcal{Q}_\Sigma, \mathcal{F})$ such that $\zeta \notin k[X_0]$. A contradiction is expected. From this assumption there is a G -valuation $v \in \mathcal{U}$ such that $v(\zeta) < 0$. By Lemma 2.6, one can find a B -eigenfunction $\xi \in A_M^{(B)}$ such that (1) $v(\xi) = v(\zeta) < 0$, and (2) $w(\xi) \geq w(\zeta) \geq 0$ for all $w \in C(\mathcal{D}) \cap \mathcal{Q}_\Sigma \sqcup \mathcal{F}$. So Condition (2) implies that $\xi \in A(\Gamma, \mathcal{D})$. Since $k[X_0]^U = A(\Gamma, \mathcal{D})$ the function ξ verifies $w(\xi) \geq 0$ for all $w \in C \cap \mathcal{Q}_\Sigma$ which contradicts (1). This shows the equality $k[X_0] = R(C(\mathcal{D}) \cap \mathcal{Q}_\Sigma, \mathcal{F})$.

Let us check that the pair $(\mathcal{D}, \mathcal{F})$ is a colored polyhedral divisor. The only non-trivial verification is to show that $(\text{Tail}(\mathcal{D}), \mathcal{F})$ is a colored cone, i.e., that $\varrho(\mathcal{F})$ does not contains 0. But this latter is a consequence of Condition (i) of Lemma 2.5.

It remains to show that for a simple G -model X of $\text{Mod}_G(\mathcal{X})$ one can construct a good B -chart such that $X = G \cdot X_0$. By the previous step, one can find a B -chart $X'_0 := X'_0(\mathcal{D}, \mathcal{F})$ corresponding to a colored polyhedral divisor $(\mathcal{D}, \mathcal{F})$ such that $X = G \cdot X'_0(\mathcal{D}, \mathcal{F})$. Let $\mathcal{F}_1 \subseteq \mathcal{F}$ be the subset of colors that does not contain any G -orbit. Then by [Kno94, 2.2], [Tim00, Proposition 2], the divisor $D_1 = \sum_{D \in \mathcal{F}_1} D$ is a globally generated Cartier divisor on X . Thus by [GL73, Theorem E], the complement $X_0 := X'_0 \setminus D_1$ is proper over an affine variety. Moreover, X_0 is a quasi-affine open subset of X'_0 , hence X_0 is affine. Now it is clear that X_0 generates X and from the construction above that it is represented by a colored polyhedral divisor $(\mathcal{D}', \mathcal{F}')$ where any color of \mathcal{F}' contains a G -orbit. This finishes the proof of the theorem. \square

The following result allows to recover the proper polyhedral divisor \mathcal{D} from its associated algebra $A(\Gamma, \mathcal{D})$.

Proposition 2.10. *Given a proper polyhedral divisor \mathcal{D} on Γ , the hypercone $C(\mathcal{D})$ is equal to*

$$\{[v_Y, a, b] \in \mathcal{Q} \mid Y \subseteq \text{Loc}(\mathcal{D}), v_Y(f) + b\langle m, a \rangle \geq 0 \text{ for all homogeneous } f \otimes \chi^m \in A(\Gamma, \mathcal{D})\}.$$

Proof. Let us denote by C_1 the set in the right-hand side. We directly have $C(\mathcal{D}) \subseteq C_1$ by definition of the associated algebra of the polyhedral divisor \mathcal{D} . Let $v = [v_Y, a, b]$ be a valuation in C_1 . If $b = 0$, then for any $m \in \sigma^\vee \cap M$ such that there is a homogeneous element $f \otimes \chi^m$ we have $\langle m, a \rangle = v(f \otimes \chi^m) \geq 0$. Since \mathcal{D} is proper, we deduce that $a \in \sigma^\vee$ and therefore $v \in C(\mathcal{D})$.

Now assume that $b \neq 0$. Dividing by b we may assume that $b = 1$. We shall extend the argument of the proof of [LPR, Lemma 4.4] in higher complexity. We define a new polyhedral divisor \mathcal{D}' (defined over $\text{Loc}(\mathcal{D})$) by letting $\mathcal{D}'_{Y'} = \mathcal{D}_{Y'}$ if $Y' \subseteq \Gamma$ is a prime divisor different from Y and $\mathcal{D}'_Y = Q + \sigma$ for Q the convex hull of $\{a\} \cup \mathcal{D}_Y$. Since $v \in C_1$, we have $A(\Gamma, \mathcal{D}') = A(\Gamma, \mathcal{D})$. Now we use the key argument of the proof of [AH06, Lemma 9.1]. Let $m \in \sigma^\vee \cap M$ such that $\mathcal{D}(m)$ and $\mathcal{D}'(m)$ are integral Weil divisors. For $y \in \Gamma$, let

$$\mathcal{D}_y(m) = \sum_{Y \subseteq \Gamma, y \in Y} \min_{v \in \mathcal{D}_Y} \langle m, v \rangle \cdot Y \text{ and } \mathcal{D}'_y(m) = \sum_{Y \subseteq \Gamma, y \in Y} \min_{v \in \mathcal{D}'_Y} \langle m, v \rangle \cdot Y.$$

Since $\mathcal{D}(m)$ is semiample, there exist $d \in \mathbb{Z}_{>0}$ and a section $f \in H^0(\text{Loc}(\mathcal{D}), \mathcal{O}_{\text{Loc}(\mathcal{D})}(d\mathcal{D}(m)))$ such that $\text{div}_y(f) + d\mathcal{D}_y(m) = 0$. Moreover, f is a global section of $\mathcal{O}_{\text{Loc}(\mathcal{D})}(d\mathcal{D}'(m))$ and so $\text{div}_y(f) + d\mathcal{D}'_y(m) \geq 0$. This implies that $\mathcal{D}(w) \leq \mathcal{D}'(w)$ for any $w \in \sigma^\vee$. Hence $\mathcal{D}'_Y \subseteq \mathcal{D}_Y$, yielding $v \in C(\mathcal{D})$. This completes the proof of the proposition. \square

Next, we show that a B -chart of $\text{Mod}_G(\mathcal{X})$ associated with a colored polyhedral divisor $(\mathcal{D}, \mathcal{F})$ does not change if we modify the locus of \mathcal{D} by a birational projective morphism.

Proposition 2.11. *Let Γ, Γ' be two semiprojective models of S with a projective birational morphism $\psi : \Gamma' \rightarrow \Gamma$. Consider a colored polyhedral divisor $(\mathcal{D}, \mathcal{F}) \in \text{CPDiv}(\Gamma, \mathcal{S})$ and denote by $\psi^*(\mathcal{D})$ the polyhedral divisor defined by the equality $\psi^*(\mathcal{D})(m) = \psi^*(\mathcal{D}(m))$ for any $m \in \text{Tail}(\mathcal{D})^\vee$. Then $(\psi^*(\mathcal{D}), \mathcal{F}) \in \text{CPDiv}(\Gamma', \mathcal{S})$ and we have*

$$X_0(\mathcal{D}, \mathcal{F}) = X_0(\psi^*(\mathcal{D}), \mathcal{F}) \text{ in } \text{Mod}_G(\mathcal{X}).$$

Proof. Since ψ is a projective birational morphism, we have the equality

$$H^0(\Gamma, \mathcal{O}_\Gamma(\mathcal{D}(m))) = H^0(\Gamma', \psi^* \mathcal{O}_\Gamma(\mathcal{D}(m))) = H^0(\Gamma', \mathcal{O}_{\Gamma'}(\psi^* \mathcal{D}(m)))$$

for any $m \in \text{Tail}(\mathcal{D})^\vee$ such that $\mathcal{D}(m)$ is an integral Cartier divisor. Hence by normality, we have $A(\Gamma, \mathcal{D}) = A(\Gamma', \psi^* \mathcal{D})$. The equality $k[X_0(\mathcal{D}, \mathcal{F})] = k[X_0(\psi^*(\mathcal{D}), \mathcal{F})]$ follows from [Tim11, Lemma 19.12]. The fact that $\psi^*(\mathcal{D})$ is proper is a consequence of [AH06, Example 8.4(i)]. \square

2.2. Localization of colored polyhedral divisors. In this subsection, we study and classify certain open immersions between B -charts of $\text{Mod}_G(\mathcal{X})$. Our idea is first to restrict to the case of a localization with the respect to a B -eigenfunction. As in the case of \mathbb{T} -varieties (see [AHS08, Sections 3, 4]), we characterize this operation in the language of (colored) polyhedral divisors.

Unlike the toric case, the B -equivariant open immersions of B -charts are not in general determined by one localization. Hence we need to study open covering given by several localizations (see Proposition 2.16). In the sequel, we will denote by $(\mathcal{D}, \mathcal{F}) \in \text{CPDiv}(\Gamma, \mathcal{S})$ a colored polyhedral divisor with tail the polyhedral cone $\sigma \subseteq N_{\mathbb{Q}}$.

Definition 2.12. Let $w \in \sigma^\vee \cap M$ and let $f \in H^0(\text{Loc}(\mathcal{D}), \mathcal{O}_{\text{Loc}(\mathcal{D})}(\mathcal{D}(w)))$ be a global section. The *localization* of $(\mathcal{D}, \mathcal{F})$ with the respect to the homogeneous element $f \otimes \chi^w$ is the colored polyhedral divisor $(\mathcal{D}, \mathcal{F})_f$ (or $(\mathcal{D}, \mathcal{F})_f^w$ if we want to specify the degree of f) defined as follows. First of all, for any σ -polyhedron $C \subseteq N_{\mathbb{Q}}$ we denote

$$\text{face}(C, w) = \{v \in C \mid \langle w, v \rangle \leq \langle w, v' \rangle \text{ for all } v' \in C\} \subseteq N_{\mathbb{Q}};$$

it is a $\sigma \cap w^\perp$ -polyhedron. One can also denote by Γ_f the subset $\Gamma \setminus Z(f)$, where

$$Z(f) = \text{supp}(\text{div}(f) + \mathcal{D}(w))$$

is the *zero locus* of the section f . Then considering

$$\mathcal{D}_f = \sum_{Y \subseteq \Gamma_f} \text{face}(\mathcal{D}_Y, w) \cdot Y \text{ and } \mathcal{F}^w = \{D \in \mathcal{F} \mid \varrho(D) \in w^\perp\},$$

the colored polyhedral divisor $(\mathcal{D}, \mathcal{F})_f$ is the pair $(\mathcal{D}_f, \mathcal{F}^w)$. Moreover, the symbol $\mathcal{D}_y := \sum_{Y \subseteq \Gamma, y \in Y} \mathcal{D}_Y \subseteq N_{\mathbb{Q}}$ will stand for the *fiber polyhedron* over the point $y \in \Gamma$.

As expected, the localization of a colored polyhedral divisor translates geometrically into the usual localization of the corresponding B -eigenfunction.

Lemma 2.13. *Let $X_0 = X_0(\mathcal{D}, \mathcal{F})$ be the associated B -chart and let $\xi = f \otimes \chi^w$ be a homogeneous element as before. Then the localization $(\mathcal{D}, \mathcal{F})_f$ describes the open B -equivariant immersion $(X_0)_\xi \rightarrow X_0$.*

Proof. By [AHS08, Proposition 3.3] we know that $k[X_0]_\xi^U = k[X_0(\mathcal{D}, \mathcal{F})_f]^U$. Hence letting f_1/ξ^r be in $k[X_0]_\xi$ with $f_1 \in k[X_0] \setminus \{0\}$, we have $v(f_1/\xi^r) \geq 0$ for any G -valuation v in $C(\mathcal{D}_f)$ (by using [Tim11, Lemma 19.12]). Since $\mathcal{F}^w \subseteq \mathcal{F}$, we also have

$$v_D(f_1/\xi) = v_D(f_1) - \langle w, \varrho(D) \rangle = v_D(f_1) \geq 0$$

for any $D \in \mathcal{F}^w$. This implies that $k[X_0]_\xi \subseteq k[X_0(\mathcal{D}, \mathcal{F})_f]$.

For the opposite inclusion, we remark that the B -chart $(X_0)_\xi$ can be described by a colored polyhedral divisor $(\mathcal{D}_f, \mathcal{F}')$ over Γ_f and it remains to establish the equality $\mathcal{F}' = \mathcal{F}^w$. Using the preceding step, we have $\mathcal{F}' \subseteq \mathcal{F}$. Therefore

$$D \in \mathcal{F}' \iff \forall r \in \mathbb{Z}, v_D(f^r) \geq 0 \iff \langle w, \varrho(D) \rangle = 0 \iff D \in \mathcal{F}^w,$$

finishing the proof of the lemma. \square

The next definition is an extension of the terminology introduced in [AHS08, Definition 4.1] for torus actions.

Definition 2.14. Let X_0, X'_0 be two B -charts of $\text{Mod}_G(\mathcal{X})$ and let $X'_0 \hookrightarrow X_0$ be a B -equivariant open immersion. A family of elements $\alpha_1, \dots, \alpha_r \in k[X_0]^U \subseteq A_M$ reduces X_0 to X'_0 if we have $X'_0 = \bigcup_{i=0}^r (X_0)_{\alpha_i}$, all the α_i 's are homogeneous and they are invertible on some \mathbb{T} -orbit closure in $\text{Spec } k[X_0]^U$, where \mathbb{T} is the algebraic torus with character lattice M .

The technical reason to considering good B -charts among others of a given G -model of \mathcal{X} (see Definition 2.9) is justified by the following lemma where the B -equivariant open immersions of B -charts are described by the unions of localizations of B -eigenfunctions. See [Tim97, Theorem 4.1 (1)] for a version of this statement in the context of reductive group actions of complexity one. We note that the proof uses results stated in Section 4. The attentive reader may remark that there is no logical issues to proceed in this way.

Lemma 2.15. *For any B -equivariant open immersion $X'_0 \hookrightarrow X_0$ of B -charts of $\text{Mod}_G(\mathcal{X})$ with X'_0 good, there exists a finite family of $k[X_0]^U$ which reduces X_0 to X'_0 (see Definition 2.14).*

Proof. Let us denote by $I \subseteq k[X_0]$ the vanishing ideal of $D = X_0 \setminus X'_0$. We may assume that $D \neq \emptyset$. Let $J \subseteq k[X_0]$ be the radical of the ideal generated by $I^{(B)}$, where $I^{(B)}$ stands for the set of B -eigenfunctions that are in I . The first aim is to show that the equality $I = J$ holds. We start by considering the open subsets $X = G \cdot X_0$ and $X' = G \cdot X'_0$, and the parabolic subgroups P, P' stabilizing the B -charts X_0 and X'_0 , respectively. Using Theorem 4.2, there exists a P -isomorphism (resp. P' -isomorphisms) $X_0 \simeq P_u \times X_\star$ (resp. $X'_0 \simeq P'_u \times X'_\star$). We refer to Lemma 4.1 for the definition of X_\star and X'_\star . Denote by $\tilde{X}_\star = \text{Spec } \mathcal{A}_\star$ the relative spectrum over Γ of the sheaf

$$\mathcal{A}_\star = \bigoplus_{\lambda \in \sigma \vee \cap M} \mathcal{O}_\Gamma(\mathcal{D}(\lambda)) \otimes_k V_\lambda,$$

where $(\mathcal{D}, \mathcal{F})$ is a colored polyhedral divisor describing X_0 (see Theorem 2.8), σ is the tail of \mathcal{D} and Γ is the locus of \mathcal{D} . We introduce similar notations for X'_\star . Then by using a similar argument as in the proof of Proposition 4.5, we have the natural proper birational P -equivariant (resp. P' -equivariant) morphisms $q : \tilde{X}_0 \rightarrow X_0$ and $q' : \tilde{X}'_0 \rightarrow X'_0$, where $\tilde{X}_0 = P_u \times \tilde{X}_\star$ and $\tilde{X}'_0 = P'_u \times \tilde{X}'_\star$. Moreover, we also have the quotient map by the P' -action $\pi' : \tilde{X}'_0 \rightarrow \Gamma$.

By Lemma 4.3, there exists a unique closed P' -orbit O_y inside each fiber $(\pi')^{-1}(y)$ which is obtained as the intersection of a closed G -orbit of $\tilde{X}' = G \cdot \tilde{X}'_0$ with \tilde{X}'_0 . Denote by ν_y the G -valuation centered in the generic point of O_y . By picking an arbitrary regular function $\alpha_y \in k[X_0]$ such that $\nu_y(\alpha_y) = 0$ and $\alpha_y(X_0 \setminus X'_0) = \{0\}$ (here the condition $\nu_y(\alpha_y) = 0$ may be understood as α_y does not vanish at the generic point of the closed P -orbit $q'(O_y)$), one can find (according to Lemma 2.6) a homogeneous

element $a_y \in A(\Gamma, \mathcal{D}) \subseteq k[X_0]$ such that $\nu_y(a_y) = 0$ and $a_y(X_0 \setminus X'_0) = \{0\}$. Indeed, we note that since X'_0 is affine the complement of X'_0 in X_0 is pure of codimension one and so the condition $a_y(X_0 \setminus X'_0) = \{0\}$ can be expressed in terms of colors and G -valuations.

Claim: we have $I = \sqrt{(a_y | y \in \Gamma)}$. Seeing a_y as a regular function on \tilde{X}_0' , we only need to check that for any $z \in (\pi')^{-1}(y)$ we have $a_y(z) \neq 0$. Indeed, this would imply that the $(\tilde{X}_0')_{a_y}$'s cover \tilde{X}_0' and therefore that the $(X'_0)_{a_y}$'s cover X'_0 . We need to describe the algebra of regular functions of the irreducible components of $(\pi')^{-1}(y)$ by adapting the description of [AH06, Proposition 7.10] in our setting. First, we recall that \mathcal{D}_y denotes the fiber polyhedron $\sum_{Y \subseteq \Gamma, y \in Y} \mathcal{D}_Y$. We also consider its normal quasifan Λ_y (see the comments before [AH06, Definition 7.1]) and the fiber monoid

$$S_y = \{\lambda \in \sigma^\vee \cap M \mid \mathcal{D}(\lambda) \text{ is a principal divisor at } y\}.$$

Using the proof of [AH06, Proposition 7.10], we may construct a sequence of natural morphisms of P' -algebras

$$\Phi : k[X'_0] \rightarrow k[(\pi')^{-1}(y)] \simeq k[P'_u] \otimes_k k[\Lambda_y, S_y],$$

where the multiplication of the algebra $k[\Lambda_y, S_y] = \bigoplus_{\lambda \in S_y} V_\lambda$ is defined as follows. For a Levi subgroup G'_\star of P' , denote by Ω'_\star the general G'_\star -homogeneous spherical orbit of X'_\star . If λ, λ' belong to the same cone $\omega \in \Lambda_y$, then the multiplication map $V_\lambda \otimes_k V_{\lambda'} \rightarrow k[\Omega'_\star]$ is the multiplication coming from the algebra $k[\Omega'_\star]$ and equal to 0 otherwise. The irreducible components of $(\pi')^{-1}(y)$ correspond to the maximal cones of Λ_y . In particular, by taking the invariant by the maximal unipotent subgroup $U \cap G'_\star$, they are normal and therefore spherical G'_\star -varieties (see [Tim11, Theorem D.5(3)]).

We now use the hypothesis of goodness for the B -chart X'_0 . By Proposition 2.21, there exists a central G'_\star -valuation v on $k[X_\star]$ which is positive on each $B_{G'_\star}$ -eigenfunction of $k[X_\star]$, where $B'_\star = B \cap G'_\star$. Therefore using the map Φ , it induces a G'_\star -valuation on each subalgebra $A_\omega = \bigoplus_{\lambda \in \omega \cap S_y} V_\lambda$ (for $\omega \in \Lambda_y$ maximal) which is positive in any B'_\star -eigenfunction. In particular the closed P' -orbit O_y determines a G'_\star -valuation w_ω in the relative interior of the colored cone of the spherical variety $\text{Spec } A_\omega$. Since the restriction of a_y on each irreducible component $P_u \times \text{Spec } A_\omega$ is nonzero on the common part O_y , the function a_y can be seen as a B -eigenfunction on each of them. We finally conclude that a_y is nonzero on any point of $(\pi')^{-1}(y)$ (by remarking that its weight is orthogonal to each of the w_ω 's). This gives the claim and establishes the equality $I = J$. Now using [AHS08, Remark 4.2], we may choose a finite family of $I^{(B)}$ generating I such that it reduces X_0 to X'_0 . This finishes the proof of the lemma. \square

The next proposition determines the 'face relations' of the colored polyhedral divisors corresponding to B -equivariant open immersions of good B -charts.

Proposition 2.16. *Let $(\mathcal{D}, \mathcal{F}), (\mathcal{D}', \mathcal{F}') \in \text{CPDiv}(\Gamma, \mathcal{S})$ be two colored polyhedral divisors with tails σ, σ' living in $N_{\mathbb{Q}}$ and loci Γ_1, Γ'_1 , respectively. Assume that $\Gamma'_1 \subseteq \Gamma_1$, $C(\mathcal{D}') \subseteq C(\mathcal{D})$, $\mathcal{F}' \subseteq \mathcal{F}$ and that $X_0(\mathcal{D}', \mathcal{F}')$ is a good B -chart. Then the induced B -equivariant dominant morphism $X_0(\mathcal{D}', \mathcal{F}') \rightarrow X_0(\mathcal{D}, \mathcal{F})$ is an open immersion if and only if for any $y \in \text{Loc}(\mathcal{D}')$ there exist $w \in \sigma^\vee \cap M$ and a section $f \in H^0(\Gamma_1, \mathcal{O}_{\Gamma_1}(\mathcal{D}(m)))$ such that $(\mathcal{F}')^w = \mathcal{F}^w$ and*

$$y \in (\Gamma_1)_f \subseteq \Gamma'_1, \mathcal{D}'_y = \text{face}(\mathcal{D}_y, w), \text{face}(\mathcal{D}'_z, w) = \text{face}(\mathcal{D}_z, w), \text{ for any } z \in (\Gamma_1)_f (\star).$$

Proof. We will use the same notation as in the proof of Lemma 2.15. We consider the natural proper birational P -equivariant (resp. P' -equivariant) morphisms $q : \tilde{X}_0 \rightarrow X(\mathcal{D}, \mathcal{F})$ and $q' : \tilde{X}_0' \rightarrow X_0(\mathcal{D}', \mathcal{F}')$. Let $\pi' : \tilde{X}_0' \rightarrow \text{Loc}(\mathcal{D}')$ be the quotient map and choose a point $y \in \text{Loc}(\mathcal{D}')$. Suppose that we have an open embedding

$$X_0(\mathcal{D}', \mathcal{F}') \rightarrow X_0(\mathcal{D}, \mathcal{F}).$$

Considering the subset $X_0(\mathcal{D}', \mathcal{F}')$ obtained by localizations of many B -eigenfunctions (see Lemma 2.15), we deduce that the natural map $X_0(\mathcal{D}', \mathcal{F}')//U \rightarrow X_0(\mathcal{D}, \mathcal{F})//U$ is a \mathbb{T} -equivariant open immersion.

Using the proof of Lemma 2.15 there exist $w \in \sigma^\vee \cap M$ and a section $f \in H^0(\Gamma_1, \mathcal{O}_{\Gamma_1}(\mathcal{D}(m)))$ such that $f \otimes \chi^w$ is non-zero on the fiber $\pi^{-1}(y)$ and identically zero on the boundary $X_0(\mathcal{D}, \mathcal{F}) \setminus X_0(\mathcal{D}', \mathcal{F}')$. Hence by the beginning of the proof [AHS08, Proposition 3.4] and considering the \mathbb{T} -equivariant open immersion $X_0(\mathcal{D}', \mathcal{F}')//U \hookrightarrow X_0(\mathcal{D}, \mathcal{F})//U$, we obtain that $w \in \sigma^\vee \cap M$ and $f \in H^0(\Gamma_1, \mathcal{O}_{\Gamma_1}(\mathcal{D}(m)))$ satisfy Condition (\star) . In addition,

$$X_0(\mathcal{D}, \mathcal{F})_{f \otimes \chi^w} = X_0(\mathcal{D}', \mathcal{F}')_{f \otimes \chi^w}.$$

Using Lemma 2.13 and [Tim11, Proposition 13.7(1)], we have $(\mathcal{F}')^w = \mathcal{F}^w$. This shows the first direction.

Let us prove the converse. For any $y \in \text{Loc}(\mathcal{D}')$ denote by $\alpha_y = f_y \otimes \chi^{w_y}$ the corresponding homogeneous element. According to the end of the proof of [AHS08, Proposition 3.4], the condition $\mathcal{D}'_y = \text{face}(\mathcal{D}_y, w)$ implies that α_y is nonzero on $(\pi')^{-1}(y)/U$. Hence remarking that α_y is constant on the U -orbits of \tilde{X}'_0 , it follows that $(\pi')^{-1}(y) \subseteq (\tilde{X}'_0)_{\alpha_y}$ for any $y \in \text{Loc}(\mathcal{D}')$. Thus we have

$$\tilde{X}'_0 = \bigcup_{y \in \text{Loc}(\mathcal{D}')} (\tilde{X}'_0)_{\alpha_y} \text{ and therefore } X_0(\mathcal{D}', \mathcal{F}') = \bigcup_{y \in \text{Loc}(\mathcal{D}')} X_0(\mathcal{D}', \mathcal{F}')_{\alpha_y}.$$

Moreover, our assumption implies that the localizations $(\mathcal{D}', \mathcal{F}')_{f_y}^{w_y}$ and $(\mathcal{D}, \mathcal{F})_{f_y}^{w_y}$ coincide. This finally gives by Lemma 2.13 the open inclusions

$$X_0(\mathcal{D}', \mathcal{F}') = \bigcup_y X_0(\mathcal{D}', \mathcal{F}')_{\alpha_y} \subseteq \bigcup_y X_0(\mathcal{D}, \mathcal{F})_{\alpha_y} = X_0(\mathcal{D}, \mathcal{F}),$$

as required. \square

Note that in the sequel we will write by the symbol $(\mathcal{D}', \mathcal{F}') \hookrightarrow (\mathcal{D}, \mathcal{F})$ for specifying that the conditions for the colored polyhedral divisors $(\mathcal{D}', \mathcal{F}'), (\mathcal{D}, \mathcal{F})$ of the preceding proposition are satisfied, i.e., $X_0(\mathcal{D}', \mathcal{F}')$ is a good B -chart and we have in particular an open immersion of B -charts $X_0(\mathcal{D}', \mathcal{F}') \hookrightarrow X_0(\mathcal{D}, \mathcal{F})$. We finish this subsection by the following two technical results which follow from an adaptation of those in [AHS08, Section 4].

Lemma 2.17. *Let $(\mathcal{D}, \mathcal{F}) \in \text{CPDiv}(\Gamma, \mathcal{S})$ be a colored polyhedral divisor on a normal semiprojective variety Γ_1 with tail $\sigma \subseteq N_{\mathbb{Q}}$. Let $X'_0 \subseteq X_0 := X_0(\mathcal{D}, \mathcal{F})$ be an inclusion of B -charts with X'_0 good. Denote by \tilde{X} the relative spectrum over Γ_1 of the sheaf*

$$\mathcal{A} = \bigoplus_{m \in \sigma^\vee \cap M} \mathcal{O}_{\Gamma_1}(\mathcal{D}(m)).$$

Consider the contraction map $r : \tilde{X} \rightarrow X_0//U$ and the quotient morphism $\pi : \tilde{X} \rightarrow \Gamma_1$. Then

$$\Gamma'_1 := \pi(r^{-1}(X'_0//U)) \subseteq \Gamma_1$$

is open and semiprojective. For a family $(\alpha_i = f_i \otimes \chi^{w_i}, i \in I)$ which reduces $X_0(\mathcal{D}, \mathcal{F})$ to X'_0 we let

$$\mathcal{D}' = \sum_{Y \subseteq \Gamma'_1} \mathcal{D}'_Y \cdot Y, \text{ where } \mathcal{D}'_Y = \bigcup_{Y \cap (\Gamma'_1)_{f_i} \neq \emptyset} \text{face}(\mathcal{D}_Y, w_i).$$

Finally, we denote by \mathcal{F}' the subset $\bigcup_{i \in I} \mathcal{F}^{w_i}$. Then the pair $(\mathcal{D}', \mathcal{F}')$ is a colored polyhedral divisor which describes the open immersion $X'_0 \subseteq X_0(\mathcal{D}, \mathcal{F})$.

Proof. The fact that Γ'_1 is open and semiprojective, and that $(\mathcal{D}', \mathcal{F}')$ is a colored polyhedral divisor is a consequence of [AHS08, Proposition 4.3]. We recall that the existence of the family $(\alpha_i)_{i \in I}$ comes from Lemma 2.15. Using the fact that $(\alpha_i)_{i \in I}$ reduces $X_0(\mathcal{D}, \mathcal{F})$ to X'_0 and the results in [AHS08, Section 4] we have

$$(X'_0)_{\alpha_i} // U = \text{Spec } A(\Gamma, \mathcal{D}'_{f_i}) = \text{Spec } A(\Gamma, \mathcal{D}_{f_i}) = (X_0)_{\alpha_i} // U \text{ for any } i.$$

Thus, the equalities $(\mathcal{F}')^{w_i} = \mathcal{F}^{w_i}$ and Lemma 2.13 give $X'_0 = \bigcup_{i \in I} (X'_0)_{\alpha_i} = \bigcup_{i \in I} (X_0)_{\alpha_i} \subseteq X_0$, as required. \square

The colored polyhedral divisor $(\mathcal{D}', \mathcal{F}')$ defined in Lemma 2.17 will be denoted by $\bigcup_{i \in I} (\mathcal{D}, \mathcal{F})_{\alpha_i}$ since it determines the union $\bigcup_{i \in I} X_0(\mathcal{D}, \mathcal{F})_{\alpha_i}$. As a direct consequence, we obtain the following corollary.

Corollary 2.18. *Let X_0 be a B -chart of $\text{Mod}_G(\mathcal{X})$ built from a colored polyhedral divisor $(\mathcal{D}, \mathcal{F}) \in \text{CPDiv}(\Gamma, \mathcal{S})$ and consider two colored polyhedral divisors*

$$\left(\mathcal{D}' = \sum_{Y \subseteq \Gamma} \mathcal{D}'_Y, \mathcal{F}' \right) = \bigcup_{i \in I} (\mathcal{D}, \mathcal{F})_{\alpha_i} \text{ and } \left(\mathcal{D}'' = \sum_{Y \subseteq \Gamma} \mathcal{D}''_Y, \mathcal{F}'' \right) = \bigcup_{j \in J} (\mathcal{D}, \mathcal{F})_{\beta_j}$$

that describe B -equivariant open immersions X'_0, X_0'' of X_0 as in Lemma 2.17. Then the intersection can be computed as

$$(\mathcal{D}', \mathcal{F}') \cap (\mathcal{D}'', \mathcal{F}'') := \left(\sum_{Y \subseteq \Gamma} \mathcal{D}'_Y \cap \mathcal{D}''_Y \cdot Y, \mathcal{F}' \cap \mathcal{F}'' \right) = \left(\bigcup_{(i,j) \in I \times J} \mathcal{D}_{f_i g_j}, \bigcup_{(i,j) \in I \times J} \mathcal{F}^{w_i + y_j} \right),$$

with $\alpha_i = f_i \otimes \chi^{w_i}$ and $\beta_i = g_i \otimes \chi^{y_i}$ for all i, j . In particular, the intersection above corresponds to the B -chart $X'_0 \cap X_0''$.

2.3. Some technical lemmas. In this subsection, we collect some technical results needed for our classification problem. The following classical lemma is a consequence of [Sum74, Section 5, Theorem 3] and [Tim11, Theorems 12.11, 12.13]. It gives valuative criteria for the properness and the separateness of the integral G -schemes of type over k . See also [Sum74, Section 4, Remark 4] for an interpretation in terms of Zariski-Riemann spaces.

Lemma 2.19. *If X is an integral scheme of finite type over k with a G -action, then X is separated over k (resp. proper over k) if and only if any G -valuation ν of $k(X)$ has at most one center (resp. exactly one center) in X .*

Let us recall some notation originally introduced in [AHS08, Section 7]. Let $(\mathcal{D}, \mathcal{F}) \in \text{CPDiv}(\Gamma, \mathcal{S})$ be a colored polyhedral divisor, where Γ is a smooth semiprojective variety, and consider s a discrete valuation on the function field $k(\Gamma)$ with center a schematic point $\xi \in \Gamma$. Since Γ is smooth, any Weil divisor is locally described by a hypersurface and so for any prime $Y \subseteq \Gamma$ we denote by f_Y the local equation of Y near the schematic point ξ . The symbol $s(\mathcal{D})$ will stand for the polyhedron

$$s(\mathcal{D}) = \sum_{Y \subseteq \Gamma} s(f_Y) \cdot \mathcal{D}_Y \subseteq N_{\mathbb{Q}}.$$

If s is trivial, then $s(\mathcal{D})$ is equal to the tail of \mathcal{D} . Let $m \in \text{Tail}(\mathcal{D})^{\vee}$ such that $\mathcal{D}(m)$ is an integral Cartier divisor and let f be a local equation of $\mathcal{D}(m)$ near ξ . Note that $s(\mathcal{D})$ is constructed in a such way that $\min\langle m, s(\mathcal{D}) \rangle = s(f)$.

Lemma 2.20. *Let $(\mathcal{D}, \mathcal{F}) \in \text{CPDiv}(\Gamma, \mathcal{S})$ be a colored polyhedral divisor on a smooth semiprojective variety Γ and let $\nu = [s, p, l] \in \mathcal{Q}$ be a G -valuation on $k(\mathcal{X})$. Denote by $X = X(\mathcal{D}, \mathcal{F})$ the simple G -model of \mathcal{X} associated with $(\mathcal{D}, \mathcal{F})$. Then ν has a center in X if and only if the valuation s has center a schematic point of Γ and*

$$\begin{cases} p/l \in s(\mathcal{D}) & \text{if } l \neq 0, \\ p \in \text{Tail}(\mathcal{D}) & \text{if } l = 0. \end{cases}$$

Proof. Let $X_0 = X_0(\mathcal{D}, \mathcal{F})$. We start by using a similar argument as in the proof of [AHS08, Lemma 7.7]. Assume that s has a center $\zeta \in X$. Since ζ is the generic point of a G -cycle X_1 , we have $X_0 \cap X_1 \neq \emptyset$ and this implies that $k[X_0] \subseteq \mathcal{O}_{\nu}$. Hence restricting the valuation ν to $k(\Gamma_0)$, where $\Gamma_0 := \text{Spec } k[\Gamma]$, and considering the projective morphism $\Gamma \rightarrow \Gamma_0$, we conclude by the valuative criterion of properness that $s = \nu|_{k(\Gamma)}$ has a center in Γ . Moreover, the inclusion $A(\Gamma, \mathcal{D}) = k[X_0]^U \subseteq \mathcal{O}_{\nu}$ and [AHS08, Lemma 7.7] imply that $p/l \in s(\mathcal{D})$ if $l \neq 0$ and $p \in \text{Tail}(\mathcal{D})$ otherwise.

Let us show the converse. By loc. cit. we have directly that $A(\Gamma, \mathcal{D}) \subseteq \mathcal{O}_{\nu}$. Now using [Tim11, Lemma 19.12], for any $f \in k[X_0]$ there exists a B -eigenfunction $\alpha \in k[X_0]^{(B)} \subseteq A(\Gamma, \mathcal{D})$ such that $\nu(f) = \nu(\alpha) \geq 0$. Hence ν has a center in X_0 and therefore in X . This completes the proof of the lemma. \square

The following lemma gives an explicit description of the goodness condition and define the "goodification" for B -charts which consists to remove the bad colors. This process is a crucial step for introducing our definition of a colored divisorial fan (see Definition 2.22).

Proposition 2.21. *Let $X_0 = X_0(\mathcal{D}, \mathcal{F})$ be a B -chart corresponding to a colored polyhedral divisor $(\mathcal{D}, \mathcal{F}) \in \text{CPDiv}(\Gamma, \mathcal{S})$. If X_0 is not good, then the relative interior of $C(\mathcal{D})$ does not intersect the hypercone \mathcal{Q}_{Σ} . Moreover, in this case, if \mathcal{F}_1 denotes the set of colors of $X = X(\mathcal{D}, \mathcal{F})$ that does not contain any G -orbit, then $D \in \mathcal{F}_1$ if and only if $\varrho(D)$ does not belong to the hyperface generated by $C(\mathcal{D}) \cap \mathcal{Q}_{\Sigma}$. The hypercone generated by $C(\mathcal{D}) \cap \mathcal{Q}_{\Sigma}$ and $\mathcal{F} \setminus \mathcal{F}_1$ defines a colored polyhedral divisor denoted by $(\mathcal{D}, \mathcal{F})^g := (\mathcal{D}^g, \mathcal{F}^g)$ on Γ and X_0^g is a good B -chart of X .*

Proof. We start by proving the first claim. Let us take $v = [s, p, l] \in \mathcal{Q}_\Sigma \cap C(\mathcal{D})$ with $s = v_Y$ for some prime divisor $Y \subseteq \Gamma$ so that $s(\mathcal{D}) = \mathcal{D}_Y \cdot Y$. Hence v is centered in the generic point of a G -cycle $Z \subseteq X(\mathcal{D}, \mathcal{F})$ (see Lemma 2.20). Now assume that v belongs to the relative interior of $C(\mathcal{D})$ (i.e., (p, l) belongs to the relative interior of the Cayley cone $C_Y(\mathcal{D})$). If $f \in A(\Gamma, \mathcal{D})$ is a homogeneous element such that $v(f) = 0$, then $w(f) = 0$ for any $w \in [v_Y, C_Y(\mathcal{D})]$ and therefore $v_D(f) = 0$ for any $D \in \mathcal{F}$. By [Tim11, Theorem 14.2] we have $Z \subseteq D$ for any $D \in \mathcal{F}$ and so X_0 is good. This shows the first claim.

We assume that X_0 is not good. For the second point, let us consider a prime divisor $Y \subseteq \text{Loc}(\mathcal{D})$ and denote by $Z_Y \subseteq X$ a G -cycle with associated G -valuation v in the relative interior of $C_Y(\mathcal{D}) \cap \mathcal{Q}_\Sigma$. Let $Q_Y \subseteq C_Y(\mathcal{D})$ be the face generated by the subset

$$\{(\alpha, \beta) \in C_Y(\mathcal{D}) \mid [v_Y, \alpha, \beta] \in C(\mathcal{D}) \cap \mathcal{Q}_\Sigma\}.$$

Since Z_Y is not contained in $\bigcup_{D \in \mathcal{F}_1} D$ there exists a homogeneous element $a \in A(\Gamma, \mathcal{D})$ such that $v(a) = 0$ and $v_D(a) > 0$ for any $D \in \mathcal{F}_1$ (compare with [Tim11, Theorem 14.2]). In addition, v is in the relative interior of Q_Y , and so $\varrho(D) \notin Q_Y$. Thus, $\varrho(D)$ does not belong to the hyperface generated by $C(\mathcal{D}) \cap \mathcal{Q}_\Sigma$.

Let us show the converse implication. Let us consider a color $D \in \mathcal{F}$ such that $\varrho(D)$ does not belong to the hyperface generated by $C(\mathcal{D}) \cap \mathcal{Q}_\Sigma$. Assume that there is a G -cycle $Z \subseteq X$ contained in D . Let $v = [s, p, l]$ be a G -valuation centered in the generic point of Z . By pulling back \mathcal{D} by a projective birational morphism if necessary (see Proposition 2.11), we may assume that $s = v_Y$ for some prime divisor $Y \subseteq \text{Loc}(\mathcal{D})$. Hence by Lemma 2.20, we have $v \in C(\mathcal{D})$. Now using Proposition 2.10, there exists a homogeneous element $a \in A(Y, \mathcal{D})$ which determines the face $Q_Y \subseteq C_Y(\mathcal{D})$, i.e., for any $w \in [v_Y, C_Y(\mathcal{D})]$, we have $w(a) = 0$ if and only if $w \in [v_Y, Q_Y]$. In particular, by Lemma 2.7, it follows that $\varrho(D) \in Q_Y$, yielding a contradiction.

Let us show the last statement. The only thing to check is that the resulting polyhedral divisor \mathcal{D}^g obtained by the hypercone generated by $C(\mathcal{D}) \cap \mathcal{Q}_\Sigma$ and $\mathcal{F} \setminus \mathcal{F}_1$ is proper over $\text{Loc}(\mathcal{D})$. Let $\tau = \text{Tail}(\mathcal{D}^g)$ and $\sigma = \text{Tail}(\mathcal{D})$. Since τ is a face of σ , there exists $m \in \sigma^\vee \cap \tau^\perp$ such that $\tau^\vee = \sigma^\vee + \mathbb{Q}m$. Moreover, denote by $V(\mathcal{D}_Y)$ the set of vertices of \mathcal{D}_Y where Y is a prime divisor of Γ . Remark that for such subsets we have the inclusions $V(\mathcal{D}_Y) \subseteq \tau$. Thus, if $w \in \tau^\vee$ with $w = m_1 + m_2$ and $m_1 \in \sigma^\vee, m_2 \in \mathbb{Q}m$, then we obtain that

$$\mathcal{D}^g(w) = \sum_{Y \subseteq \Gamma} \min_{v \in V(\mathcal{D}_Y)} \langle m_1 + m_2, v \rangle \cdot Y = \sum_{Y \subseteq \Gamma} \min_{v \in \mathcal{D}_Y} \langle m_1, v \rangle \cdot Y = \mathcal{D}(m_1).$$

In addition, for a given w in the relative interior τ^\vee , we may choose m_1 to be in the relative interior of σ^\vee . Therefore the above formula implies that \mathcal{D}^g is proper. This finishes the proof of the proposition. \square

2.4. Colored divisorial fans. We now describe the G -models of \mathcal{X} , that is, all normal G -varieties that are G -birational to $\mathcal{X} = S \times \Omega$ in terms of geometrico-combinatorial objects which we will call colored divisorial fans. In the special case of torus actions on normal varieties, this class of objects restricts to those of divisorial fans introduced in [AHS08, Definition 5.2] and comprises the defining fans of toric varieties.

Without loss of generality, we will assume that all polyhedral divisors are defined on a smooth semiprojective variety Γ (where Γ is a model of S) by taking a pull back by a projective resolution of singularities (compare with Proposition 2.11) if necessary. We recall that \mathcal{S} denotes the homogeneous spherical datum of the spherical G -homogeneous space Ω .

Here we give the definition of a colored divisorial fan.

Definition 2.22. A *colored divisorial fan* associated with the pair (Γ, \mathcal{S}) is a finite set

$$\mathcal{E} = \{(\mathcal{D}^i, \mathcal{F}^i) \in \text{CPDiv}(\Gamma, \mathcal{S}) \mid i \in I\}$$

of colored polyhedral divisors satisfying the following properties.

- (i) The intersections $(\mathcal{D}^i \cap \mathcal{D}^j, \mathcal{F}^i \cap \mathcal{F}^j)$, for $i, j \in I$, belong to \mathcal{E} , where

$$\mathcal{D}^i \cap \mathcal{D}^j := \sum_{Y \subseteq \Gamma} \mathcal{D}_Y^i \cap \mathcal{D}_Y^j \cdot Y,$$

and \mathcal{E} is stable by goodification (i.e., if $(\mathcal{D}, \mathcal{F}) \in \mathcal{E}$, then $(\mathcal{D}, \mathcal{F})^g \in \mathcal{E}$, see Proposition 2.21 for an explicit description).

- (ii) For all i, j , we have the open immersions $(\mathcal{D}^i \cap \mathcal{D}^j, \mathcal{F}^i \cap \mathcal{F}^j)^g \hookrightarrow (\mathcal{D}^i, \mathcal{F}^i)$, i.e., the natural maps

$$X_0(\mathcal{D}^i, \mathcal{F}^i) \leftarrow X_0(\mathcal{D}^i \cap \mathcal{D}^j, \mathcal{F}^i \cap \mathcal{F}^j)^g \rightarrow X_0(\mathcal{D}^j, \mathcal{F}^j)$$

are open immersions (see Proposition 2.16 for a geometrico-combinatorial description).

- (iii) Let $\sigma_{ij} \subseteq N_{\mathbb{Q}}$ be the tail cone of $\mathcal{D}^i \cap \mathcal{D}^j$. Then we have

$$\mathcal{F}^i \cap \mathcal{F}^j = \varrho^{-1}(\sigma_{ij}) \cap \mathcal{F}^i = \varrho^{-1}(\sigma_{ij}) \cap \mathcal{F}^j$$

for all $i, j \in I$, where $\varrho: \mathcal{F}_{\Omega} \rightarrow \mathbb{Z}$ is the coloration map defined in Section 1.2.

- (iv) For any geometric valuation μ on the function field $k(\Gamma) = k(S)$ we have

$$\mu(\mathcal{D}^i) \cap \mu(\mathcal{D}^j) \cap \mathcal{V} = \mu(\mathcal{D}^i \cap \mathcal{D}^j) \cap \mathcal{V}$$

for all $i, j \in I$, where \mathcal{V} is the valuation cone of the spherical homogeneous space Ω .

We may define the *locus* of \mathcal{E} as

$$\text{Loc}(\mathcal{E}) := \bigcup_{(\mathcal{D}, \mathcal{F}) \in \mathcal{E}} \text{Loc}(\mathcal{D}) \subseteq \Gamma.$$

Remark 2.23. Note that the *face relations* in Definition 2.22 (ii) can be expressed in the language of linear systems. For simplicity, denote by $(\mathcal{D}', \mathcal{F}')$ the intersection $(\mathcal{D}^i \cap \mathcal{D}^j, \mathcal{F}^i \cap \mathcal{F}^j)^g$ and by $(\mathcal{D}, \mathcal{F})$ the colored polyhedral divisor $(\mathcal{D}^i, \mathcal{F}^i)$ for fixed indexes $i, j \in J$.

The relation $(\mathcal{D}', \mathcal{F}') \hookrightarrow (\mathcal{D}, \mathcal{F})$ holds if and only if for any prime divisor $Y \subseteq \Gamma$ we have $\mathcal{D}'_Y \subseteq \mathcal{D}_Y$ and for any $y \in \Gamma$ there exist $w_y \in \text{Tail}(\mathcal{D}')^{\vee} \cap M$ and Y_y in the linear system $|\mathcal{D}(w_y)|$ such that (see Proposition 2.16):

- (i) The point y does not belong to the support of Y_y ;
- (ii) The fiber polyhedron \mathcal{D}'_y (see Definition 2.12) is equal to $\text{face}(\mathcal{D}_y, w_y)$;
- (iii) The equality $\text{face}(\mathcal{D}'_z, w_z) = \text{face}(\mathcal{D}_z, w_z)$ holds for any point $z \in \Gamma$ outside the support of Y_z , and:
- (iv) We have $(\mathcal{F}')^{w_y} = \mathcal{F}^{w_y}$ for any $y \in \Gamma$.

In particular, the polyhedron \mathcal{D}'_Y is a face of \mathcal{D}_Y for any prime divisor $Y \subseteq \Gamma$. We emphasize that such relations are only available a priori in the case where the B -chart $X_0(\mathcal{D}', \mathcal{F}')$ is good. More precisely, the family

$$\{\mathcal{D}_y \cap \mathcal{V} \mid (\mathcal{D}, \mathcal{F}) \in \mathcal{E}\}$$

forms a polyhedral subdivision in the valuation cone \mathcal{V} for any point $y \in \Gamma$ (but probably not outside \mathcal{V}).

In the next example, we explain how we recover the Luna-Vust theory for spherical embeddings (see [Kno91]) via the language of colored divisorial fans.

Example 2.24. *Spherical varieties.* In the case where $k(\mathcal{X})^B = k$, the G -models of \mathcal{X} are exactly the usual embeddings of the spherical homogeneous space Ω and a colored divisorial fan \mathcal{E} consists in particular of a finite set of colored cones. The face relations between two elements $(\sigma', \mathcal{F}'), (\sigma, \mathcal{F}) \in \mathcal{E}$ with $(\sigma', \mathcal{F}')^g = (\sigma', \mathcal{F}')$ are given by the expected conditions that σ' is a face of σ and $\mathcal{F}' = \mathcal{F} \cap \varrho^{-1}(\sigma')$. By adding elements if necessary, we may assume that \mathcal{E} is stable by face relations. Hence the set $\Theta = \{(\sigma, \mathcal{F})^g \mid (\sigma, \mathcal{F}) \in \mathcal{E}\}$ defines a colored fan in the sense of [Kno91, Section 3]. Its elements are in one to one correspondence with the G -orbits of the corresponding embedding $X_{\Theta} \supseteq \Omega$.

We can now enunciate the main result of this subsection.

Theorem 2.25. *Let \mathcal{E} be a colored divisorial fan on (Γ, \mathcal{S}) , where Γ is a smooth projective model of the variety S and \mathcal{S} is the homogeneous spherical datum of the spherical G -homogeneous space Ω . Then the open subscheme*

$$X(\mathcal{E}) := \bigcup_{(\mathcal{D}, \mathcal{F}) \in \mathcal{E}} X(\mathcal{D}, \mathcal{F}) \subseteq \text{Mod}_G(\mathcal{X})$$

is a G -model of $\mathcal{X} = S \times \Omega$ in which the open subset $X(\mathcal{D} \cap \mathcal{D}', \mathcal{F} \cap \mathcal{F}')$ is identified with the intersection $X(\mathcal{D}, \mathcal{F}) \cap X(\mathcal{D}', \mathcal{F}')$ for all $(\mathcal{D}, \mathcal{F}), (\mathcal{D}', \mathcal{F}') \in \mathcal{E}$. Conversely, any G -model of \mathcal{X} arises in this way.

Proof. Let $\mathcal{E} = \{(\mathcal{D}^i, \mathcal{F}^i)\}_{i \in I}$ be a colored divisorial on (Γ, \mathcal{S}) . We start by proving the first claim, namely that the open subscheme $X(\mathcal{E})$ in $\text{Mod}_G(\mathcal{X})$ is obtained by gluing charts $X(\mathcal{D}, \mathcal{F})$ for $(\mathcal{D}, \mathcal{F}) \in \mathcal{E}$, where their intersections are given by the intersections of colored polyhedral divisors. We first observe that we have the commutative diagram

$$\begin{array}{ccccc} X_0^j & \xleftrightarrow{\quad} & X_0(\mathcal{D}^i, \mathcal{F}^i) & \xleftarrow{\quad} & X_0^l \\ & \searrow & \uparrow & \swarrow & \\ & & X_0(\mathcal{D}^i \cap \mathcal{D}^j, \mathcal{F}^i \cap \mathcal{F}^j)^g \cap X_0(\mathcal{D}^i \cap \mathcal{D}^l, \mathcal{F}^i \cap \mathcal{F}^l)^g & & \end{array}$$

Here X_0^j and X_0^l are defined by the equalities

$$X_0^e = X_0(\mathcal{D}^i \cap \mathcal{D}^j, \mathcal{F}^i \cap \mathcal{F}^e)^g$$

for $e = j, l$. Note that the horizontal maps are the open immersions given by the definition of a divisorial fan. Moreover, by Corollary 2.18 the other maps are also open immersions.

The natural morphisms

$$X_0(\mathcal{D}^i, \mathcal{F}^i) \leftarrow X_0(\mathcal{D}^i \cap \mathcal{D}^j, \mathcal{F}^i \cap \mathcal{F}^j)^g \rightarrow X_0(\mathcal{D}^j, \mathcal{F}^j)$$

induces the open immersions

$$X(\mathcal{D}^i, \mathcal{F}^i) \leftarrow X(\mathcal{D}^i \cap \mathcal{D}^j, \mathcal{F}^i \cap \mathcal{F}^j) \rightarrow X(\mathcal{D}^j, \mathcal{F}^j)$$

which we denote respectively by η_{ij} and η_{ji} . Put $X_{ij} = \eta_{ij}(X(\mathcal{D}^i \cap \mathcal{D}^j))$. To see this maps as a gluing, we need to check the following cocycle conditions:

$$\varphi_{ij}(X_{ij} \cap X_{il}) = X_{ji} \cap X_{jl} \text{ and } \varphi_{il} = \varphi_{jl} \circ \varphi_{ij},$$

where φ_{ij} is the composition $\eta_{ji} \circ \eta_{ij}^{-1}$. Since the maps φ_{ij} are inclusion of open subsets in the scheme $\text{Mod}_G(\mathcal{X})$, it suffices to show that

$$G \cdot (X_0(\mathcal{D}^i \cap \mathcal{D}^j, \mathcal{F}^i \cap \mathcal{F}^j)^g \cap X_0(\mathcal{D}^i \cap \mathcal{D}^l, \mathcal{F}^i \cap \mathcal{F}^l)^g) = X(\mathcal{D}^i \cap \mathcal{D}^j \cap \mathcal{D}^l, \mathcal{F}^i \cap \mathcal{F}^j \cap \mathcal{F}^l).$$

But by Corollary 2.18, we have

$$\begin{aligned} & X_0(\mathcal{D}^i \cap \mathcal{D}^j, \mathcal{F}^i \cap \mathcal{F}^j)^g \cap X_0(\mathcal{D}^i \cap \mathcal{D}^l, \mathcal{F}^i \cap \mathcal{F}^l)^g \\ &= X_0((\mathcal{D}^i \cap \mathcal{D}^j)^g \cap (\mathcal{D}^i \cap \mathcal{D}^l)^g, (\mathcal{F}^i \cap \mathcal{F}^j)^g \cap (\mathcal{F}^i \cap \mathcal{F}^l)^g) \text{ and therefore} \\ & X_0((\mathcal{D}^i \cap \mathcal{D}^j)^g \cap (\mathcal{D}^i \cap \mathcal{D}^l)^g, (\mathcal{F}^i \cap \mathcal{F}^j)^g \cap (\mathcal{F}^i \cap \mathcal{F}^l)^g) \\ &= X_0(\mathcal{D}^i \cap \mathcal{D}^j \cap \mathcal{D}^l, \mathcal{F}^i \cap \mathcal{F}^j \cap \mathcal{F}^l)^g. \end{aligned}$$

Hence the open subset $X(\mathcal{E})$ is an integral scheme of finite type over k in which the open subset $X(\mathcal{D} \cap \mathcal{D}', \mathcal{F} \cap \mathcal{F}')$ is identified with the intersection $X(\mathcal{D}, \mathcal{F}) \cap X(\mathcal{D}', \mathcal{F}')$ for all $(\mathcal{D}, \mathcal{F}), (\mathcal{D}', \mathcal{F}') \in \mathcal{E}$.

We now show that $X(\mathcal{E})$ is separated over k by using Condition (iv) of Definition 2.22 and Lemma 2.19. We follow the argument of the proof of [AHS08, Proposition 7.5]. Let $\nu = [s, p, l]$ be a G -valuation on $k(\mathcal{X})$ having centers the schematic points ξ and ξ' in $X(\mathcal{E})$. We may assume that $l \neq 0$. Then ξ, ξ' belong respectively to some dense open G -stable subsets $X(\mathcal{D}, \mathcal{F})$ and $X(\mathcal{D}', \mathcal{F}')$, where $(\mathcal{D}, \mathcal{F}), (\mathcal{D}', \mathcal{F}') \in \mathcal{E}$. By Lemma 2.20, the restriction $s = \nu|_{k(\Gamma)}$ has a unique center in Γ and

$$p \in s(\mathcal{D}) \cap s(\mathcal{D}') \cap \mathcal{V} = s(\mathcal{D} \cap \mathcal{D}') \cap \mathcal{V}.$$

This implies by Lemma 2.20 that ν has a center in $X(\mathcal{D} \cap \mathcal{D}', \mathcal{F} \cap \mathcal{F}')$. Since $X(\mathcal{D}, \mathcal{F})$ and $X(\mathcal{D}', \mathcal{F}')$ are separated, we obtain that $\xi = \xi'$. By Lemma 2.19, we conclude that the subscheme $X(\mathcal{E})$ is a G -model of \mathcal{X} .

Conversely, let us consider X a G -model of \mathcal{X} . By the Sumihiro theorem (see [Sum74, Theorem 1 and Lemma 8], [Kno91, Theorem 1.3]) there exists a G -stable open covering $(X_i)_{i \in I}$ of X by simple G -varieties, where I is a finite set and $X_i \subseteq \text{Mod}_G(\mathcal{X})$ for any $i \in I$. By Theorem 2.8, each X_i is described by a colored polyhedral divisor $(\mathcal{D}^i, \mathcal{F}^i) \in \text{CPDiv}(\Gamma_i, \mathcal{S})$ on a normal semiprojective variety $\Gamma_i \subseteq \text{Mod}(S)$. We follow the argument of the proof of [AHS08, Theorem 5.6]. Let $\bar{\Gamma}_i$ be a projective compactification of Γ_i such that the complement $\bar{\Gamma}_i \setminus \Gamma_i$ is the support of a semiample divisor. In this way, any colored polyhedral divisor $(\mathcal{D}^i, \mathcal{F}^i)$ is defined on $\bar{\Gamma}_i$ by adding empty coefficients if necessary. Moreover the inclusions $X_i \cap X_j \subseteq X_i$ induce birational maps between $\bar{\Gamma}_i$ and $\bar{\Gamma}_j$. By resolving the indeterminacies and using the Hironaka theorem, we obtain a smooth projective variety $\Gamma \subseteq \text{Mod}(S)$

which dominates all the $\bar{\Gamma}_i$'s and is compatible with the initial rational maps. Then using Lemma 2.17 and Proposition 2.11, and adding the goodified colored polyhedral divisors, we conclude that the pull backs to Γ of all the colored polyhedral divisors $(\mathcal{D}^i, \mathcal{F}^i)$'s form a set \mathcal{E} satisfying Conditions (i), (ii), (iii) of Definition 2.22.

It remains to show that \mathcal{E} verifies the Condition 2.22 (iv). Let us assume that this condition does not hold for \mathcal{E} . Then there exist $(\mathcal{D}, \mathcal{F}), (\mathcal{D}', \mathcal{F}') \in \mathcal{E}$ and a geometric valuation μ on $k(S)$ such that

$$\mu(\mathcal{D} \cap \mathcal{D}') \cap \mathcal{V} \subsetneq \mu(\mathcal{D}) \cap \mu(\mathcal{D}') \cap \mathcal{V}.$$

Let $p \in \mu(\mathcal{D} \cap \mathcal{D}') \cap \mathcal{V}$ which does not belong to $\mu(\mathcal{D}) \cap \mu(\mathcal{D}') \cap \mathcal{V}$. Then by Lemma 2.20, the G -valuation $\nu = [\nu, p, 1]$ has no center in $X(\mathcal{D} \cap \mathcal{D}', \mathcal{F} \cap \mathcal{F}')$ but has center ξ, ξ' in $X(\mathcal{D}, \mathcal{F})$ and $X(\mathcal{D}', \mathcal{F}')$, respectively. Since by the preceding steps we have

$$X(\mathcal{D}, \mathcal{F}) \cap X(\mathcal{D}', \mathcal{F}') = X(\mathcal{D} \cap \mathcal{D}', \mathcal{F} \cap \mathcal{F}'),$$

we conclude that $\xi \neq \xi'$, which contradicts the separateness of X and completes the proof of the theorem. \square

Our next task is to give a criterion of completeness for the G -models of \mathcal{X} in terms of its defining colored divisorial fan. For this purpose, we introduce the notion of completeness for colored divisorial fans.

Let \mathcal{E} be a colored divisorial fan on (Γ, \mathcal{S}) . We say that \mathcal{E} is *complete* if Γ is smooth projective and for any geometric valuation s on $k(\Gamma)$ we have $\bigcup_{(\mathcal{D}, \mathcal{F}) \in \mathcal{E}} s(\mathcal{D}) \cap \mathcal{V} = \mathcal{V}$, where \mathcal{V} is the valuation cone of the spherical homogeneous space Ω .

Proposition 2.26. *Let \mathcal{E} be colored divisorial fan defining a G -model X of \mathcal{X} . Then X is complete if and only if \mathcal{E} is complete.*

Proof. We will use the criterion of Lemma 2.19. The completeness of X provided by the assumption \mathcal{E} complete is a direct consequence of Lemma 2.20. Let us assume that X is complete. From loc. cit. we have Y projective and if

$$E := \bigcup_{(\mathcal{D}, \mathcal{F}) \in \mathcal{E}} s(\mathcal{D}) \cap \mathcal{V} \subsetneq \mathcal{V},$$

then choosing a vector $p \in \mathcal{V} \setminus E$ and according to Lemma 2.19, the G -valuation $\nu = [s, p, 1]$ on $k(\mathcal{X})$ has a center in X . This gives a contradiction and completes the proof of the proposition. \square

2.5. Explicit construction. This subsection aims to construct explicitly the simple G -variety associated with a colored polyhedral divisor $(\mathcal{D}, \mathcal{F})$ via an embedding into a projective space. We will follow the idea of the proof of [Kno91, Theorem 3.1].

We start by taking homogeneous generators $h_i = f_i \otimes \chi^{m_i}$ ($1 \leq i \leq r$) of the M -graded algebra $A(\Gamma, \mathcal{D})$, where every f_i is in $k(\Gamma)^*$. The functions χ^{m_i} considered as B -eigenfunctions on $k(\Omega)$ have their poles contained in the subset

$$Z_0 = \bigcup_{D \in \mathcal{F}_\Omega \setminus \mathcal{F}} D \subseteq \Omega.$$

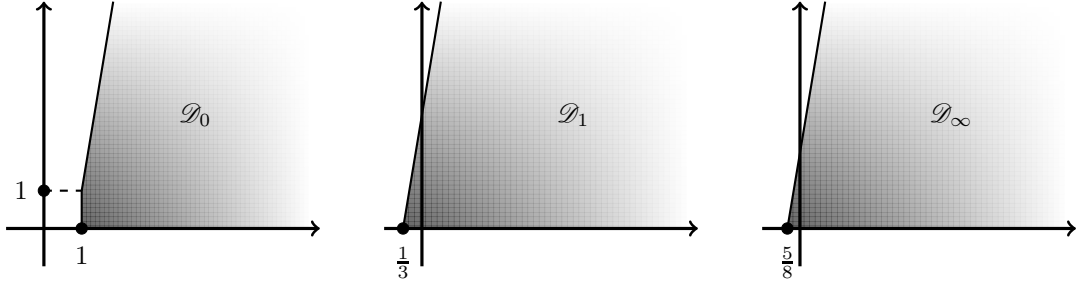
Hence using that G is factorial (since simply-connected), we may choose a function $\xi_0 \in k[G]^{(B \times H)}$ with zero locus equal to $\pi^{-1}(Z_0)$ and such that $\xi_i := \xi_0 \cdot h_i \in k[G]$ for $1 \leq i \leq r$, where $\pi : G \rightarrow \Omega$ is the natural projection.

Let V be the G -module generated by $\xi_0, \xi_1, \dots, \xi_r$ in $k(\Gamma) \otimes_k k[G]$. We finally obtain a natural G -equivariant rational map

$$\iota : \Gamma \times \Omega \dashrightarrow \mathbb{P}(V^\vee).$$

Assuming that the colored polyhedral divisor $(\mathcal{D}, \mathcal{F})$ gives rise to a good B -chart, we have the following theorem.

Theorem 2.27. *Let $X_0 = \bar{X} \cap \{\xi_0 \neq 0\}$ where \bar{X} is the closure of the image of the map ι and let $X = G \cdot X_0 \subseteq \mathbb{P}(V^\vee)$. The G -variety X is G -isomorphic to $X(\mathcal{D}, \mathcal{F})$ and the B -chart $X_0(\mathcal{D}, \mathcal{F})$ is identified with X_0 .*



Proof. Let \hat{X} be the affine cone arising from $\bar{X} \hookrightarrow \mathbb{P}(V^\vee)$. The grading on $k[\hat{X}]$ is given by $\bigoplus_{d \in \mathbb{N}} R_d$, where R_d is the subvector space generated by monomials in elements of V of degree d . Moreover, we have $k[X_0] = \{f/\xi_0^d \mid f \in R_d, d \in \mathbb{N}\}$. From this we deduce that $k[X_0]^U = A(\Gamma, \mathcal{D})$ (adapt the argument of [Kno91, Theorem 3.1]) and by [Tim00, §3, Lemma 1] that X_0 (and so X) is normal. Moreover, the rational map ι is induced by the inclusion $k[X_0] \subseteq k(\Gamma) \otimes_k k(\Omega)$. Since \mathcal{D} is proper, the map ι is birational. Hence X is described by a colored polyhedral divisor $(\mathcal{D}, \mathcal{F}')$ and it remains to show that $\mathcal{F} = \mathcal{F}'$. Note that, by construction, the hyperplane $\xi_0 = 0$ does not contain any G -orbit. Hence we have $\mathcal{F} \subseteq \mathcal{F}'$. By goodness, one can find a G -valuation $\nu = [\cdot, p, 0]$ on the function field $k(\mathcal{X})$ with p in the relative interior of $\text{Tail}(\mathcal{D})$ (compare with Proposition 2.21). Then by Lemma 2.20, ν has a center $\xi \in X$. Assume that $\xi \notin D$ for some $D \in \mathcal{F}'$. By [Tim11, Lemma 19.12] and using that $X_0 \cap D \neq \emptyset$, one can find $f \in k[X_0]^{(B)}$ such that $v(f) = 0$ and $v_D(f) > 0$. But the fact that p is in the interior relative of $\text{Tail}(\mathcal{D})$ implies that $v_{D'}(f) = 0$ for all $D' \in \mathcal{F}'$, which gives a contradiction and establishes $\mathcal{F} = \mathcal{F}'$. \square

Example 2.28. In this example $G = \text{SL}_2 \times \text{SL}_2$. We consider the spherical G -homogeneous space $\Omega = \text{SL}_2/U \times \text{SL}_2/U$, where U is a maximal unipotent subgroup of SL_2 . Here $M = N = \mathbb{Z}^2$. We have exactly two colors D_1, D_2 in Ω which are sent respectively on the first and second vectors of the canonical basis. We denote by $(\mathcal{D}, \mathcal{F})$ the colored polyhedral divisor defined by the equalities $\mathcal{D} = \mathcal{D}_0 \cdot [0] + \mathcal{D}_1 \cdot [1] + \mathcal{D}_\infty \cdot [\infty]$ and $\mathcal{F} = \{D_1\}$. The locus of \mathcal{D} is \mathbb{P}^1 and

$$\mathcal{D}_0 = [(1, 0), (1, 1)] + \sigma, \quad \mathcal{D}_1 = \left(-\frac{1}{3}, 0\right) + \sigma, \quad \mathcal{D}_\infty = \left(-\frac{5}{8}, 0\right) + \sigma,$$

with $\sigma = \mathbb{Q}_{\geq 0}(1, 0) + \mathbb{Q}_{\geq 0}(1, 24)$ (see the figure). Using the natural G -action on $\mathbb{A}^2 \times \mathbb{A}^2$, the irreducible representations inside $k[\Omega]$ are of the form $V(\lambda, \nu) = E(\lambda) \otimes_k F(\nu)$ ($\lambda, \nu \in \mathbb{Z}_{\geq 0}$) where

$$E(\lambda) = \bigoplus_{i+j=\lambda, i, j \geq 0} kx_0^i x_1^j \text{ and } F(\nu) = \bigoplus_{i+j=\nu, i, j \geq 0} ky_0^i y_1^j.$$

Then by [Lan13, Example 2.6] we have the equality

$$A(\mathbb{P}^1, \mathcal{D}) = k \left[y_0, \frac{(1-z)^8}{z^{23}} x_0^{24} y_0^{-1}, \frac{1-z}{z^3} x_0^3, \frac{(1-z)^3}{z^8} x_0^8 \right] \simeq k[u, v, w, t]/(uv - w^8 + t^3).$$

Here z is a local coordinate of \mathbb{P}^1 , i.e., $k(\mathbb{P}^1) = k(z)$. The subvariety $y_0 = 0$ in Ω corresponds to the color D_2 . Denote by V the G -submodule generated by $y_0^2, y_0 u, y_0 v, y_0 w, y_0 t$ in $k(z) \otimes_k k(\Omega)$. Then \bar{X} is the Zariski closure of the image of the morphism

$$\mathbb{A}^1 \setminus \{0\} \times \Omega \rightarrow \mathbb{P}(V^\vee), \quad (z, [M_1], [M_2]) \mapsto [\phi_{z, [M_1], [M_2]}],$$

where the linear form $\phi_{z, [M_1], [M_2]}$ is the usual evaluation function on the triple $(z, [M_1], [M_2])$. Finally, by regarding y_0 as element of the bidual $V \simeq V^{\vee\vee}$, the complement of $\{y_0 = 0\}$ in \bar{X} corresponds to the chart $X_0(\mathcal{D}, \mathcal{F})$.

3. CLASSIFICATION

Let X be a normal G -variety with spherical orbits. In this section, we show that up to a modification by a (global) G -equivariant Galois covering the G -variety X is G -isomorphic (on a dense open subset) to a product $S \times \Omega$, where $\Omega = G/H$ is a spherical homogeneous space and S is a variety in which G acts trivially, see Theorem 3.6.

The proof uses a result of Alexeev and Brion (cf. [AB05, Theorem 3.1]) which implies that X has a general stabilizer. Thus, to obtain our result we adapt the argument of the proof of a result of Colliot-Thélène, Kunyavskii, Popov and Reichstein which describes the equivariant birational type of any G -variety with a general stabilizer. We refer to [PV89, Paragraph 2.5] for an abstract description of the G -equivariant birational type of a G -variety in term of the relative Galois cohomology. Note that some results were obtained for the case of certain affine G -varieties of complexity one, see for instance [Arz97, Section 3, Proposition 3] and [AB06, Proposition 3.12]. Finally, we state our classification result for normal G -varieties with spherical orbits in Theorem 3.14.

3.1. Describing the G -equivariant birational type. Our starting point is the following result.

Theorem 3.1. [AB05, Theorem 3.1] *Let X be a G -variety with spherical orbits. Then there exist a closed spherical subgroup $H \subseteq G$ and a G -stable dense open subset $X_1 \subseteq X$ such that any isotropy group of a point of X_1 is conjugated to H .*

The homogeneous space $\Omega = G/H$ in the above statement will be called *the general orbit* of X . The reader is referred to [Ric72] for various results on the existence of a unique general orbit for reductive group actions. We will also use later on the following lemma.

Lemma 3.2. *Let F be a finite group acting on an integral scheme X of finite type over k . Then the following assertions hold.*

(i) *The F -action is faithful if and only if it is generically free.*

For the next points, assume that any F -orbit of X is contained in an affine open subset. For instance, this applies for the case where X is covered by F -stable quasi-projective open subsets.

(ii) *The F -scheme X admits a good categorical quotient $\pi : X \rightarrow X'$, where $X' = X/F$ is an integral scheme of finite type over k .*

(iii) *If the F -action is free, then the quotient map $\pi : X \rightarrow X'$ is an étale morphism.*

(iv) *The field extension $k(X)/k(X')$ is Galois with Galois group F if and only if the F -action on X is generically free.*

Proof. Assertion (i) follows from a classical argument: the complement of the subset in which the F -action is free is the finite union of closed subsets $\bigcup_{F_1} X^{F_1}$, where F_1 runs all subgroups of F of cardinality > 1 . Assertion (ii) is a consequence of [Mum70, Page 111, III, Theorem 1 (A)]. For Assertion (iii), the fact that $\pi : X \rightarrow X'$ is a finite flat morphism is explained in [Mum70, Page 112, III, Theorem 1 (B)]. Finally, the morphism π is unramified since we work over a base field of characteristic zero. Let us show Assertion (iv). Since the extension $k(X)/k(X')$ is separable, the number of points of a general fiber of $X \rightarrow X'$ is $[k(X) : k(X')]$. Hence the F -action is generically free if and only if the cardinality of F is $[k(X) : k(X')]$. This finishes the proof of the lemma. \square

The following is a classical result which asserts that a variety with an action of a connected linear algebraic group admits a *quasi-section*.

Lemma 3.3. [PV89, Proposition 2.7] *Let X be a variety with an action of a connected linear algebraic group E . Then after changing X by an E -stable dense open subset, there exists an irreducible reduced closed subset where the intersection with any E -orbit is a 0-dimensional closed subscheme.*

The next lemma is a straightforward observation but useful for the sequel. It determines the H -fixed point set of the homogeneous space G/H .

Lemma 3.4. *Let Ω be a homogeneous space G/H . Then we have the equality*

$$\Omega^H = \{gH \mid g \in N_G(H)\} = N_G(H)/H$$

and for any $x \in \Omega^H$ the isotropy group G_x is equal to H .

Proof. We have $gH \in \Omega^H \Leftrightarrow HgH = gH \Leftrightarrow g \in N_G(H)$. Moreover, if $x = gH \in \Omega^H$, then $h \in G_x \Leftrightarrow hgH = gH \Leftrightarrow H = gHg^{-1} = h^{-1}H \Leftrightarrow h \in H$, finishing the proof of the lemma. \square

By a *Galois covering* with Galois group F we mean a dominant finite morphism $\pi : X \rightarrow X'$ such that the field extension $k(X)/k(X')$ is Galois with Galois group F . In particular, π can be viewed as the quotient of X by the generically free action (compare with Lemma 3.2 (iv)) of F . The following classical

theorem describes the birational type of any G -variety having a general stabilizer in terms of its rational quotient and its general orbit (see [CKPR11, Theorem 2.13]). Since the conclusion of the statement in *loc. cit.* is slightly different than ours, for the convenience of the reader we include here a short proof.

Theorem 3.5. [CKPR11, Theorem 2.13] *Let \mathcal{X}' be a G -variety having a general G -orbit $\Omega = G/H$ (possibly not spherical). Then there exist a variety S and a G -equivariant rational map*

$$\gamma : \mathcal{X} := S \times \Omega \dashrightarrow \mathcal{X}'$$

which is a finite Galois covering on a G -stable dense open subset. Let F be the Galois group. After shrinking S if necessary, the map γ is constructed in a such way that it induces a morphism

$$S \rightarrow \mathcal{X}', \quad s \mapsto \gamma(s, H)$$

which naturally gives a Galois covering $\tilde{\gamma} : S \rightarrow S'$ with Galois group F , where S' is a quotient of \mathcal{X}' by the algebraic group G on an open G -stable subset. The comorphism $\tilde{\gamma}^ : k(S') \rightarrow k(S)$ is induced by $\gamma^* : k(\mathcal{X}') \rightarrow k(\mathcal{X})$.*

Proof. By the Rosenlicht theorem [Ros63], [Spr89, Satz 2.2] and Theorem 3.1, there exist a G -stable dense open subset $V \subseteq \mathcal{X}'$ such that every G -orbit is G -isomorphic to Ω and a global geometric quotient $\pi_0 : V \rightarrow S'$ where S' is a d -dimensional variety. Note that the fibers of π_0 are exactly the G -orbits of V and they intersect the fixed point subscheme V^H . The natural action of $N_G(H)$ (obtained by restriction of the G -action) preserves V^H . Using the equality $V^H = \bigcup_{x \in V} (G \cdot x)^H$ and Lemma 3.4, we observe that all the isotropy groups of points of V^H for the G -action on V are equal to H . In particular the $N_G(H)$ -action on V^H factorizes into a free K -action, where $K = N_G(H)/H$. Let us take an irreducible component V_1 of V^H sending dominantly on S' by the morphism π_0 . Let K_0 be the neutral component of K . The closed subset V_1 admits a K_0 -stable dense open subset V_0 which contains an irreducible reduced closed subscheme V' intersecting any K_0 -orbit of V_0 in a finite set (see Lemma 3.3).

Let us remark that every K_0 -orbit of V_1 is contained in a fiber of π_0 and so V' is sent dominantly on S' . Thus, there exists a d -dimensional variety $S_1 \subseteq V'$ such that the restricted morphism $\pi_0 : S_1 \rightarrow S'$ is dominant; this latter being equivalent to find a closed schematic point in the fiber of the generic point $\eta \in S'$ of $\pi_0|_{V'}$. By shrinking S_1 and S' if necessary, one can find a variety S (via a Galois extension of $k(S_1)$) and a commutative diagram of finite morphisms

$$\begin{array}{ccc} S & \xrightarrow{\pi_2} & S_1 \\ & \searrow \pi_1 & \downarrow \pi_0 \\ & & S' \end{array}$$

such that π_1 is a Galois covering.

Claim 1: *The intersection of S_1 with any $N_G(H)$ -orbit of V^H is finite.* Indeed, writing $K = N_G(H)/H$ as a finite union of K_0 -cosets

$$K = \bigcup_{i=1}^l g_i K_0$$

and picking any point $x \in V^H$, we have

$$S_1 \cap N_G(H) \cdot x = S_1 \cap \left(\bigcup_{i=1}^l K_0 \cdot x_i \right) = \bigcup_{i=1}^l (S_1 \cap K_0 \cdot x_i),$$

where $x_i = g_i \cdot x$ for $i = 1, \dots, l$. Since by construction the (reduced) intersection between S_1 and each orbit $K_0 \cdot x_i$ is a finite set, we conclude that $S_1 \cap N_G(H) \cdot x$ is finite. This proves Claim 1.

Let us consider the G -equivariant map

$$\gamma : S \times \Omega \rightarrow V, \quad (s, gH) \mapsto g \cdot \pi_2(s).$$

Claim 2: *The map γ is dominant finite over an open subset.* Let $x \in V$ be an element belonging to the image of γ and consider

$$(s_1, g_1 H), (s_2, g_2 H) \in \gamma^{-1}(x).$$

If $g = g_2^{-1} g_1$, then we have $g \cdot \pi_2(s_1) = \pi_2(s_2)$. Since $\pi_2(s_2) \in V^H$ for any $h \in H$, it follows that

$$hg \cdot \pi_2(s_1) = h \cdot \pi_2(s_2) = \pi_2(s_2) = g \cdot \pi_2(s_1)$$

and $g^{-1}hg \in G_{\pi_2(s_1)} = H$ for any $h \in H$. This gives $g \in N_G(H)$. Hence there exists $u \in N_G(H)$ such that $g_2 = g_1u$ and $\pi_2(s_2) = u \cdot \pi_2(s_1)$. According to Claims 1 and the fact that K_0 acts freely on V_1 , the subset

$$Z = \{uH \in K \mid u \cdot \pi_2(s_1) \in (N_G(H) \cdot \pi_1(s_1)) \cap S_1\}$$

is finite. We finally obtain that

$$\gamma^{-1}(x) \subseteq \{(s, g_1uH) \mid s \in \pi_2^{-1}(u \cdot \pi_2(s_1)) \text{ and } uH \in Z\}$$

and γ is quasi-finite. It remains to show that γ is dominant. The image γ is a G -stable subset of V containing S_1 . Now S_1 is chosen in such way that it intersects a G -orbit of V taken in general position, hence γ is dominant. This concludes the proof of Claim 2.

Claim 3: The map γ is a G -equivariant Galois covering on a G -stable dense open subset. Indeed, the morphism γ can be obtained from the composition of the G -equivariant morphisms

$$S \times \Omega \xrightarrow{\phi_1} S_1 \times \Omega \xrightarrow{\phi_2} V,$$

$$\text{where } \phi_1(s, gH) = (\pi_2(s), gH) \text{ and } \phi_2(t, gH) = g \cdot t \text{ for all } g \in G, s \in S \text{ and } t \in S_1.$$

It induces the field extensions

$$k(V) \subseteq k(S \times \Omega) \subseteq k(S_1 \times \Omega)$$

and since γ is finite over an open subset (see Claim 2), the extension $k(S_1 \times \Omega)/k(V)$ is finite. Moreover, $k(S_1 \times \Omega)$ is Galois over $k(S \times \Omega)$ and therefore over $k(V)$. This implies Claim 3 and concludes the proof of the theorem. \square

By combining [AB05, Theorem 3.1] and the preceding result, we obtain the following theorem.

Theorem 3.6. *Let \mathcal{X}' be a G -variety with a general spherical orbit $\Omega = G/H$ and consider the finite Galois covering*

$$\gamma : \mathcal{X} = S \times \Omega \rightarrow \mathcal{X}'$$

obtained from Theorem 3.5. Let F be the Galois group acting by G -equivariant birational transformations on $\text{Mod}_G(\mathcal{X})$. Then for any G -model X' of \mathcal{X}' , there exists an F -stable G -model X of \mathcal{X} with regular F -action such that $X' = X/F$.

Proof. Let $K = k(\mathcal{X}')$ and $E = k(\mathcal{X})$. We consider a G -model X' of \mathcal{X}' . We define the variety X as the normalization of X' with the respect to the finite field extension E/K . One can construct X as follows: denote by \mathcal{A}^E the sheaf of $\mathcal{O}_{X'}$ -algebras associated with the presheaf

$$U \mapsto \overline{\mathcal{O}_{X'}(U)} \subseteq E,$$

where $\overline{\mathcal{O}_{X'}(U)}$ is the integral closure of $\mathcal{O}_{X'}(U)$ in the field extension E for any dense open subset $U \subseteq X'$. Then X is the relative spectrum $\text{Spec}_{X'} \mathcal{A}^E$. In particular, this normal scheme is finite over X' , by [EGAII, Section 1, Proposition 1.2.4], it is separated over k and is therefore a model of \mathcal{X} . Moreover, F acts regularly on X , has an open covering by F -stable affine subsets and (see Lemma 3.2 (ii)) X' is identified with the quotient X/F .

Let us show that the open subscheme $X \subseteq \text{Mod}(\mathcal{X})$ is a G -model of \mathcal{X} . We may suppose that X' is simple. Let X'_0 be a B -chart of X' intersecting any G -orbit. Its coordinate ring can be regarded as

$$k[X'_0] = \bigcap_{v \in \mathcal{Q}'} \mathcal{O}_v \cap \bigcap_{D \in \mathcal{F}' \cup D(\mathcal{X}')} \mathcal{O}_{v_D},$$

where $(\mathcal{Q}', \mathcal{F}')$ is an admissible pair as in [Tim00, Theorem 3] and $D(\mathcal{X}')$ is the set of all prime divisors on any G -model of \mathcal{X}' that are not B -stable. Note that \mathcal{Q}' is a set of G -valuations of K and \mathcal{F}' is a set of colors. Using the map γ , we deduce that the integral closure of

$$\bigcap_{D \in \mathcal{F}' \cup D(\mathcal{X}')} \mathcal{O}_{v_D} \text{ in } E \text{ is } \bigcap_{D \in \mathcal{F} \cup D(\mathcal{X})} \mathcal{O}_{v_D},$$

where \mathcal{F} consists of the irreducible components of $\gamma^{-1}(D)$ for D running through \mathcal{F}' . Moreover, a discrete valuation v on K is G -invariant if and only if any of its extension on E is G -invariant. This follows from [Tim11, Corollary 19.6] and the fact that the Galois group F acts transitively on the set of

valuations rings of E dominating \mathcal{O}_v . (compare [Mat89, Chapter 4, Exercice 12.1]). Hence by considering the set \mathcal{Q}'' of valuations extended those of \mathcal{Q}' , the integral closure of $k[X'_0]$ in E is

$$A = \bigcap_{v \in \mathcal{Q}''} \mathcal{O}_v \cap \bigcap_{D \in \mathcal{F} \cup D(\mathcal{X})} \mathcal{O}_{v_D}.$$

Claim: the pair $(\mathcal{Q}'', \mathcal{F})$ satisfies Conditions (i) and (ii) of Lemma 2.5.

Indeed, let us check Condition (i). Let E_0 be a finite subset in the disjoint union R_0 of \mathcal{Q}'' and \mathcal{F} . Then by considering the set

$$E'_0 := \{v|_K \mid v \in E_0\}$$

and since $(\mathcal{Q}', \mathcal{F}')$ verifies the conditions of [Tim00, Theorem 3], there exists a homogeneous element ξ in $\bigcap_{D \in D(\mathcal{X}')} \mathcal{O}_{v_D}$ such that $v(\xi) \geq 0$ and $w(\xi) > 0$ for all $v \in R_0$ and $w \in E'_0$. As the valuations of R_0 are obtained from the restriction of the valuations belonging to the union of \mathcal{Q}' and \mathcal{F}' , we conclude that (i) is satisfied. Finally, Condition (ii) follows from the fact that A is of finite type over k (see [Tim11, Theorem 13.8 (2)]) since it is the integral closure of $k[X'_0]$ in E . This establishes the claim. We conclude that $X_0 := \text{Spec } A$ is a B -chart of $\text{Mod}_G(\mathcal{X})$. Moreover, the subset $g \cdot X_0$ coincides with the preimage of $g \cdot X'_0$ under the natural map $X \rightarrow X'$ for any $g \in G$, and so $X = G \cdot X_0$ is a G -model of \mathcal{X} . This completes the proof of the theorem. \square

The next step is to give an interpretation of the equivariant birational classes of G -varieties with spherical orbits in term of Galois cohomology. Let S' be a variety with function field $E = k(S')$. Let us fix an algebraic closure \overline{E} of the field E . The G -isomorphism classes of forms of the G -algebra $E \otimes_k k(\Omega)$ over E are parametrized by the first pointed set of Galois cohomology

$$\mathfrak{H} := H^1(\overline{E}/E, \text{Aut}_G(\overline{E} \otimes_k k(\Omega)))$$

with coefficients in the G -equivariant automorphism group of the \overline{E} -algebra $\overline{E} \otimes_k k(\Omega)$. Writing $K := N_G(H)/H$, we have

$$\text{Aut}_G(\overline{E} \otimes_k k(\Omega)) = \text{Aut}_{G_{\overline{E}}}(G(\overline{E})/H(\overline{E})) = N_G(H)(\overline{E})/H(\overline{E}) = K(\overline{E}).$$

We note that according to [BP87, p 283] the algebraic group K is a diagonalizable group. Hence decomposing $K = K_{\text{tor}} \times K_0$ into a direct product with its torsion group K_{tor} and its neutral component K_0 , the short exact sequence

$$\{1\} \rightarrow K_0 \rightarrow K \rightarrow K_{\text{tor}} \rightarrow \{1\}$$

induces an injection in Galois cohomology

$$\{1\} = H^1(\overline{E}/E, K_0(\overline{E})) \rightarrow \mathfrak{H} \rightarrow H^1(\overline{E}/E, K_{\text{tor}}(\overline{E})),$$

where the vanishing in the left-hand side follows from the Hilbert Theorem 90. With this in hand, we obtain the following corollary.

Corollary 3.7. *The G -equivariant birational class of a G -variety with general spherical orbit $\Omega = G/H$ and the rational quotient S' determine an element of the set \mathfrak{H} , and vice-versa. Moreover, if $N_G(H)$ is connected, then \mathfrak{H} is a singleton.*

Proof. The first claim is a direct consequence of Theorem 3.6 and the above discussion. For the last claim, the connectedness of $N_G(H)$ implies that $H^1(\overline{E}/E, K_{\text{tor}}(\overline{E})) = \{1\}$ and so \mathfrak{H} is a singleton. \square

We can reformulate the last corollary for the case of G -varieties with horospherical orbits. In particular, we recover [Kno90, Satz 2].

Corollary 3.8. *Let \mathcal{X} be a G -variety with general horospherical orbit $\Omega = G/H$ and geometric quotient S on a G -stable dense open subset. Then \mathcal{X} is G -equivariantly birational to $S \times \Omega$.*

Proof. Since $N_G(H)$ is a parabolic subgroup (hence connected), we conclude that \mathfrak{H} is a singleton by using Corollary 3.7. \square

Example 3.9. *Diagonalizable matrices.* Note that, in general, a G -variety with a unique general orbit has not a trivial equivariant birational type even if all the orbits are spherical. A basic example (see [Bri96, Page 23]) is to look at the action by conjugacy of the general linear group $G = \text{GL}_n$ on the space of $n \times n$ -matrices X . Indeed, the subset of diagonalizable matrices with distinct eigenvalues forms an open

subset $X_0 \subseteq X$ where all the stabilizers are conjugated to a maximal torus T . In addition, the T -fixed point set of X_0 is irreducible while the T -fixed point set of $S \times G/T$ for any variety S is in bijection with the Weyl group of G (see Lemma 3.4), that is the symmetric group \mathfrak{S}_n with n letters. Hence the equivariant birational type of X cannot be trivial for $n > 1$.

3.2. Classification of normal G -varieties with spherical orbits. The results of the previous subsection allow us to introduce the following terminology.

Definition 3.10. Let \mathcal{X}' be a G -variety with general orbit the spherical homogeneous space Ω and rational quotient a smooth projective variety S' . By a *splitting* of \mathcal{X}' we mean the data γ of a G -equivariant generically free action of a finite abelian group F on a product variety $\mathcal{X} := S \times \Omega$ which corresponds to a non-trivial cohomology class (see Corollary 3.7) if γ is not the identity map. We also denote by the same letter γ the quotient map $\gamma : \mathcal{X} \rightarrow \mathcal{X}'$.

Note that by Theorem 3.6 and the fact that the group $\text{Aut}^G(\Omega)$ of G -equivariant automorphisms of Ω is abelian, the G -variety \mathcal{X}' admits always a splitting γ . Using the map γ , we observe that the set of colors of \mathcal{X}' is in bijection with the set of F -orbits of colors of Ω .

In order to study the G -models of \mathcal{X}' we need in particular to consider the models of S admitting a natural F -action. This leads us to state the following lemma where the proof left to the reader is similar as in [Tim11, Proposition 12.2].

Lemma 3.11. *Let S as above with its generically free regular F -action. Denote by $\text{Mod}_F(S)$ the subset of normal schematic points $\xi \in \text{Mod}(S)$ satisfying*

$$\rho^*(\mathcal{O}_{\text{Mod}(S), \xi}) \subseteq \mathcal{O}_{F \times \text{Mod}(S), (1, \xi)},$$

where ρ is the comorphism of the rational F -action on $\text{Mod}(S)$. In other words, the subset $\text{Mod}_F(S)$ is the locus of $\text{Mod}(S)$ where the F -action is regular. Then $\text{Mod}_F(S)$ is a dense open subscheme of $\text{Mod}(S)$.

In the sequel, a separated F -stable dense open subscheme of $\text{Mod}_F(S)$ of finite type over k will be called an F -model of S . As an other ingredient for defining our classification object involving in the description of G -models of \mathcal{X}' is the F -action on the set of G -valuations \mathcal{Q}_Σ . Let $\zeta = [s, a, b] \in \mathcal{Q}_\Sigma$ be a G -valuation. We denote by $g \cdot \zeta = [g \cdot s, a, b]$ the transformation of ζ under $g \in F$, where $g \cdot s$ is the valuation defined by $(g \cdot s)(f) = s(g \cdot f)$ for any $f \in k(S)$.

Let us introduce the notion of colored divisorial fans with the respect to a splitting γ of \mathcal{X}' .

Definition 3.12. A *colored divisorial fan* on $(\Gamma, \mathcal{S}, \gamma)$ is a usual colored divisorial fan \mathcal{E} on (Γ, \mathcal{S}) (see Definition 2.22) with the following properties.

- (i) The symbol Γ stands for an F -model of S which is a smooth projective variety.
- (ii) For every element $(\mathcal{D}, \mathcal{F}) \in \mathcal{E}$, the subset $C(\mathcal{D}) \cap \mathcal{Q}_\Sigma$ is F -stable for the F -action on the valuation set \mathcal{Q}_Σ .
- (iii) For every element $(\mathcal{D}, \mathcal{F}) \in \mathcal{E}$, the set of colors \mathcal{F} is F -stable, where F is seen as a subgroup of $\text{Aut}^G(\Omega)$.

Before stating our result on the classification of G -models of \mathcal{X}' , we first start with the following lemma which gives a sufficient condition for the regularity of the birational F -action on the simple G -models of \mathcal{X} .

Lemma 3.13. *Let $X = X(\mathcal{D}, \mathcal{F})$ be a simple G -model of \mathcal{X} , where $(\mathcal{D}, \mathcal{F}) \in \text{CPDiv}(\Gamma, \mathcal{S})$ is a colored polyhedral divisor. If $(\mathcal{D}, \mathcal{F})$ satisfies Conditions (i), (ii), (iii) of Definition 3.12, then the birational F -action on X is regular.*

Proof. Let u be an element of F . We will identify F with an abelian subgroup of the G -equivariant automorphism group $\text{Aut}^G(\Omega) = N_G(H)/H$ of Ω and of the automorphism group of the k -scheme $\text{Mod}_F(S)$. More precisely, u induces two automorphisms

$$\varphi_u : \Omega \rightarrow \Omega, \quad gH \mapsto guH \quad \text{and} \quad \psi_u : \text{Mod}_F(S) \rightarrow \text{Mod}_F(S), \quad s \mapsto u \cdot s.$$

Let $X_0 = X_0(\mathcal{D}, \mathcal{F})$ be the corresponding B -chart. Since $X = G \cdot X_0$ and the birational F -action on X commutes with its G -action, F -acts regularly on X if and only if it does on X_0 , or equivalently if

and only if the subalgebra $k[X_0] \subseteq k(\mathcal{X})$ is F -stable. Moreover, there exists a group homomorphism $w_u : M \rightarrow \mathbb{G}_m$ such that for any B -eigenfunction χ^m in $k(\Omega)$ of weight m we have $\varphi_u^*(\chi^m) = w_u(m)\chi^m$ (see [BP87]). Consequently, if $\xi = f \otimes \chi^m$ is a B -eigenfunction of $k(\mathcal{X})$, then u acts on ξ by the formula

$$u \cdot \xi = \psi_u^*(f) \otimes w_u(m)\chi^m.$$

Assume that $(\mathcal{D}, \mathcal{F})$ satisfies Conditions (i),(ii),(iii) of Definition 3.12. Let $f \in k[X_0] \setminus \{0\}$. If $D \in \mathcal{F}$, then Condition (iii) implies that $v_D(u \cdot f) = v_{\varphi_u(D)}(f) \geq 0$. In addition, from the discussion above, for any G -valuation $\nu = [s, v, p] \in C(\mathcal{D}) \cap \mathcal{Q}_\Sigma$ the G -valuation

$$k(\mathcal{X})^* \rightarrow \mathbb{Q}, f \mapsto \nu(u \cdot f)$$

is equal to $[u \cdot s, v, p]$ and by Conditions (i),(ii), the subalgebra $k[X_0]$ is F -stable. This finishes the proof of the lemma. \square

The following theorem completes the construction of normal G -varieties with spherical orbits in terms of the geometry of their rational quotient. Indeed, each normal G -variety X with spherical orbits has a general orbit Ω according to a result of Alexeev and Brion (see [AB05, Theorem 3.1]). The spherical homogeneous space Ω is completely described by its homogeneous spherical datum \mathcal{S} (see [Lun01, Los09, BP16]). Moreover, the G -equivariant birational type of X can be constructed via an explicit Galois covering with total space a G -equivariant trivial family $S \times \Omega$ (see Theorem 3.6) and it is determined by a Galois cohomology class (see Corollary 3.7). Finally as stated thereafter, one can construct X by a geometrico-combinatorial object \mathcal{E} depending only on the G -equivariant birational type of X .

Theorem 3.14. *Let \mathcal{X}' be a G -variety with spherical orbits. Let $\gamma : \mathcal{X} \rightarrow \mathcal{X}'$ be a splitting. Let \mathcal{E} be a colored divisorial fan on $(\Gamma, \mathcal{S}, \gamma)$. Then every local chart $X(\mathcal{D}, \mathcal{F})$ corresponding to $(\mathcal{D}, \mathcal{F}) \in \mathcal{E}$ admits a G -equivariant regular F -action coming from the rational action on \mathcal{X} . In addition,*

$$X(\mathcal{E}, \gamma) := \bigcup_{(\mathcal{D}, \mathcal{F}) \in \mathcal{E}} X(\mathcal{D}, \mathcal{F}, \gamma) \subseteq \text{Mod}_G(\mathcal{X}')$$

is a G -model of \mathcal{X}' , where $X(\mathcal{D}, \mathcal{F}, \gamma) = X(\mathcal{D}, \mathcal{F})/F$. The subset $X(\mathcal{D} \cap \mathcal{D}', \mathcal{F} \cap \mathcal{F}', \gamma)$ is identified with the intersection

$$X(\mathcal{D}, \mathcal{F}, \gamma) \cap X(\mathcal{D}', \mathcal{F}', \gamma) \text{ for all } (\mathcal{D}, \mathcal{F}), (\mathcal{D}', \mathcal{F}') \in \mathcal{E}.$$

Conversely, any G -model of \mathcal{X}' arises in this way. The variety $X(\mathcal{E}, \gamma)$ is complete if and only if \mathcal{E} is complete.

Proof. Let \mathcal{E} be a colored divisorial fan on $(\Gamma, \mathcal{S}, \gamma)$. Note that each F -variety $X(\mathcal{D}, \mathcal{F})$ is quasi-projective (see [Tim00, Section 5, Lemma 2(1)]) and therefore the quotient $X(\mathcal{D}, \mathcal{F}, \gamma)$ is well-defined. Hence by combining Theorem 2.25 and Lemmas 3.2(ii), 3.13(a), we construct the quotient space $X(\mathcal{E}, \gamma) = X(\mathcal{E})/F$ as a normal G -scheme of finite type over k which is G -birational to \mathcal{X}' . Moreover, the separateness of $X(\mathcal{E}, \gamma)$ is equivalent to the one of $X(\mathcal{E})$ (see [SGA1, Exposé V, Corollaire 1.5]). We conclude that $X(\mathcal{E}, \gamma)$ is a G -model of the G -variety \mathcal{X}' .

Conversely, let us consider a G -model X' of \mathcal{X}' and denote by $q : X \rightarrow X'$ the quotient map obtained from Theorem 3.6. Since the morphism q is affine and G -equivariant, the G -variety X admits a finite open covering of simple G -models $(X_i)_{i \in I}$ provided by $X_i = q^{-1}(X'_i)$, where X'_i is a simple G -stable dense open subset of X' . Now each open subset X_i is described by a colored polyhedral divisor $(\mathcal{D}^i, \mathcal{F}^i) \in \text{CPDiv}(\Gamma_i, \mathcal{S})$, where Γ_i is a normal semiprojective variety which is a model of S (see Theorem 2.8). Using Lemma 2.11, we may assume that each \mathcal{D}^i is minimal in the sense of [AH06, Definition 8.7]. Since $k[X_0(\mathcal{D}^i, \mathcal{F}^i)]$ is F -stable and the group F acts as well on $k[X_0(\mathcal{D}^i, \mathcal{F}^i)]^U = A(\Gamma, \mathcal{D}^i)$, by [AH06, Theorem 8.8] we deduce that Conditions (i),(ii),(iii) are satisfied for each $(\mathcal{D}^i, \mathcal{F}^i)$.

Again by resolving the indeterminacy in an F -equivariant way, we may construct a smooth projective F -model Γ in $\text{Mod}_F(S)$ that dominates all the Γ_i as in the argument of the proof of [AHS08, Theorem 5.8]. By pulling back the colored polyhedral divisors $(\mathcal{D}^i, \mathcal{F}^i)$ on Γ (see Lemma 2.11), we obtain a colored divisorial fan \mathcal{E} on $(\Gamma, \mathcal{S}, \gamma)$ such that $X = X(\mathcal{E})$. Therefore the quotient space gives $X' = X(\mathcal{E}, \gamma)$. The last claim is a direct consequence of Proposition 2.26 and the fact that the quotient map $q : X(\mathcal{E}) \rightarrow X(\mathcal{E}, \gamma)$ is a proper morphism. This finishes the proof of the theorem. \square

4. LOCAL STRUCTURE

The local structure theorem is an important result which asserts that a normal variety with an action of a connected reductive group can be expressed locally as the product of an affine space and an affine variety with an action of a Levi subgroup (see [BLV86], [Tim11, Section 4] for more information). In this section, we investigate this result for normal G -varieties with spherical orbits and translates it into the language of colored polyhedral divisors (see Theorem 4.2). As an application, we describe explicitly the invariant divisors under a Borel subgroup and give a presentation by generators and relations of the class group (see Theorem 4.6).

As usual, the symbol $\mathcal{S} = \mathcal{S}_\Omega = (\Delta^p, \Sigma, \mathbf{A}, M)$ denotes the homogeneous spherical datum of the spherical G -homogeneous space Ω .

4.1. The local structure theorem. We start by recalling the notion of localization of homogeneous spherical data (cf. [Lun97], [Lun01, Section 3.2]). We first consider a subset $\Delta_a \subseteq \Delta$ of the set of simple roots of (B, T) . Then the *localization* of \mathcal{S} with the respect to Δ_a is the datum

$$\mathcal{S}_a = (\Delta_a^p, \Sigma_a, \mathbf{A}_a, M),$$

$$\text{where } \begin{cases} \Sigma_a = \Sigma \cap \text{vect}_{\mathbb{Q}}(\Delta_a), \\ \Delta_a^p = \Delta^p \cap \Delta_a, \\ \mathbf{A}_a = \{D \in \mathbf{A} \mid \zeta(D) \cap \Delta_a \neq \emptyset\}. \end{cases}$$

The symbol $\zeta(D)$ denotes the set of roots $\alpha \in \Delta$ such that the associated minimal parabolic subgroup P_α moves D . It is a homogeneous spherical datum for the Levi subgroup of G corresponding to Δ_a . The case which we will encounter is when

$$\Delta_a = \Delta_\star := \Delta \setminus \bigcup_{D \notin \mathcal{F}} \zeta(D).$$

We denote by $\mathcal{S}_\star = (\Delta_\star^p, \Sigma_\star, \mathbf{A}_\star, M)$ the corresponding homogeneous spherical datum.

Let us explain the meaning of this operation in terms of spherical homogeneous spaces. For the parabolic subgroup P associated with Δ_\star and a Levi decomposition $P = G_\star \ltimes P_u$, the datum \mathcal{S}_\star corresponds to the spherical G_\star -homogeneous space Ω_\star satisfying the following property (see for instance [Gag15, Proposition 3.2]). Considering a colored cone (σ, \mathcal{F}) and a spherical embedding W of Ω attached to it, we may define the B -chart

$$W_0 = W \setminus \bigcup_{D \in \mathcal{F}_\Omega \setminus \mathcal{F}} \bar{D}.$$

By the local structure theorem (see [Tim11, §4.2]), we have a decomposition $P_u \times W_\star \simeq W_0$, where W_\star is a G_\star -stable subvariety of W_0 which is spherical for the acting group G_\star . The open G_\star -orbit in W_\star is G_\star -isomorphic to Ω_\star . In the sequel, we will denote by the letter \mathcal{F}_\star the set of colors of Ω_\star . Note that the spherical G_\star -variety W_\star is described by the colored cone $(\sigma, \mathcal{F}_\star)$.

In order to study the local structure for the G -variety $X = X(\mathcal{D}, \mathcal{F}, \gamma)$, we need to introduce the intermediate affine G_\star -variety X_\star . Its definition is specified in the next paragraphs. We keep the same notation as before for the spherical G_\star -homogeneous space Ω_\star .

We now consider the multiplicity-free rational G_\star -module $k[\Omega_\star] = \bigoplus_{\lambda \in \Lambda} V_\lambda$. Here $\Lambda \subseteq M$ is a saturated semigroup of dominant weights, and V_λ is the simple G_\star -submodule associated with λ . For all $\lambda, \mu \in \Lambda$ the subset $V_\lambda \cdot V_\mu$ is a direct sum of simple submodules V_ν . The possible differences $\lambda + \mu - \nu$ belong to the semigroup generated by Σ_\star [Bri91, §1.2]. More precisely, let us define the partial order \leq_Λ on Λ by letting $\mu \leq_\Lambda \lambda$ if $\lambda - \mu$ is a non-negative integral linear combination of elements of Σ_\star . Thus, we have the G -module inclusion

$$V_\lambda \cdot V_\mu \subseteq \bigoplus_{\nu \leq_\Lambda \lambda + \mu} V_\nu.$$

Lemma 4.1. *Consider a colored polyhedral divisor $(\mathcal{D}, \mathcal{F}) \in \text{CPDiv}(\Gamma, \mathcal{S})$. Then the subset*

$$A_\star(\Gamma, \mathcal{D}) := \bigoplus_{\lambda \in \text{Tail}(\mathcal{D})^\vee \cap M} H^0(\text{Loc}(\mathcal{D}), \mathcal{O}_{\text{Loc}(\mathcal{D})}(\mathcal{D}(\lambda))) \otimes_k V_\lambda \subseteq k(\Gamma) \otimes_k k(\Omega_\star)$$

is a G_\star -subalgebra. Moreover, the G_\star -scheme $X_\star = \text{Spec } A_\star(\Gamma, \mathcal{D})$ identifies with the G_\star -model of $\mathcal{X}_\star = \Gamma \times \Omega_\star$ corresponding to the colored polyhedral divisor $(\mathcal{D}, \mathcal{F}_\star)$.

Proof. For the first claim, we only need to show that if $\lambda, \mu, \nu \in \Lambda$ satisfy $\nu \leq_\Lambda \lambda + \mu$, then $\mathcal{D}(\nu) \leq \mathcal{D}(\lambda + \mu)$, i.e., $\mathcal{D}(\lambda + \mu) - \mathcal{D}(\nu)$ is an effective \mathbb{Q} -divisor. We remark that for such λ, μ, ν we have $\lambda + \mu - \nu \in \mathcal{V}_\star^\vee$, where \mathcal{V}_\star is the valuation cone of Ω_\star . Since $\Sigma_\star \subseteq \Sigma$, we have the inclusions $\mathcal{D}_Y \subseteq \mathcal{V} \subseteq \mathcal{V}_\star$ for any prime divisor $Y \subseteq \Gamma$. Hence we obtain that

$$\min_{v \in \mathcal{D}_Y} \langle \lambda + \mu, v \rangle - \min_{v \in \mathcal{D}_Y} \langle \nu, v \rangle \geq \min_{v \in \mathcal{D}_Y} \langle \lambda + \mu - \nu, v \rangle \geq 0,$$

yielding the first claim. By properness of \mathcal{D} , the scheme X_\star is a G_\star -model of \mathcal{X}_\star (see [Tim11, Theorem D5]). Finally, the colored polyhedral divisor $(\mathcal{D}, \mathcal{F}_\star)$ describes X_\star since it is a B -chart (see [Tim11, Corollary 13.10] and Theorem 2.8). \square

The next result determines the local structure of a normal G -variety with spherical orbits in terms of colored polyhedral divisors.

Theorem 4.2. *Let $P \subseteq G$ be the parabolic subgroup associated with Δ_\star . Let $P = P_u \ltimes G_\star$ be a Levi decomposition. The local structure for the simple G -variety $X = X(\mathcal{D}, \mathcal{F}, \gamma)$ can be expressed as follows. Consider the B -chart $X'_0 \subseteq X$ attached to $(\mathcal{D}, \mathcal{F}, \gamma)$ (which corresponds to the quotient of $X_0 = X_0(\mathcal{D}, \mathcal{F})$ by F). Then there exists a closed G_\star -stable subvariety $X'_\star \subseteq X'_0$ such that the map*

$$\pi_\star : P \times^{G_\star} X'_\star = P_u \times X'_\star \rightarrow X'_0, \quad (u, x) \mapsto u \cdot x$$

is a G_\star -isomorphism. We have a similar decomposition $X_0 \simeq P_u \times X_\star$ for the B -chart of $(\mathcal{D}, \mathcal{F})$ and the induced G_\star -equivariant F -action on $P_u \times X_\star$ is trivial on the first factor. The variety X_\star is the variety defined in Lemma 4.1 and $X'_\star = X_\star/F$.

Proof. Using Theorem 3.14, we shall describe the local structure for the B -chart X_0 and obtain the general case by taking the quotient by F . Consider the parabolic subgroup

$$P_1 = \{g \in G \mid g \cdot X_0 \subseteq X_0\}.$$

By Theorem 2.27, we have

$$X_0 = X(\mathcal{D}, \mathcal{F}) \setminus \bigcup_{D \in \mathcal{F}_\Omega \setminus \mathcal{F}} \bar{D},$$

where we identify colors of Ω with colors of $X(\mathcal{D}, \mathcal{F})$. Indeed, $X(\mathcal{D}, \mathcal{F})$ has a dense open subset G -isomorphic to $\Gamma_0 \times \Omega$, where $\Gamma_0 \subseteq \Gamma$ is an open subset. To any color $D \in \mathcal{F}_\Omega$ the closure \bar{D} of $\Gamma_0 \times D$ in $X(\mathcal{D}, \mathcal{F})$ defines a color and each of them is obtained in this way. Hence P_1 coincides with the parabolic subgroup P preserving $\mathcal{F}_\Omega \setminus \mathcal{F}$.

Consequently, by [Tim00, §5, Lemma 2], there exists a closed G_\star -stable subvariety $Z \subseteq X_0$ such that the map

$$\pi_\star : P \times^{G_\star} Z = P_u \times Z \rightarrow X_0, \quad (u, x) \mapsto u \cdot x$$

is a G_\star -isomorphism. Let us show that Z is G_\star -isomorphic to X_\star . Denoting by U_\star (resp. B_\star) the maximal unipotent subgroup $G_\star \cap U$ (resp. the Borel subgroup $G_\star \cap B$), we obtain that

$$K[X_0]^U = A(\Gamma, \mathcal{D}) = k[Z]^{U_\star} \text{ and } k(X_0)^B = k(\Gamma) = k(Z)^{B_\star}.$$

Moreover, by identifying the complement of the union of the colors of $\mathcal{F}_\Omega \setminus \mathcal{F}$ in Ω with the product $P_u \times \Omega_\star$ and using the map π_\star , it follows that the G_\star -variety Z has a G_\star -stable dense open subset G_\star -isomorphic to $\Gamma_0 \times \Omega_\star$. We conclude that Z is G_\star -isomorphic to X_\star .

Now we deduce our result by determining the F -action on $P_u \times Z$ via the map π_\star . Since the F -action on X_0 commutes with the P_u -action, we observe that the F -action on $P_u \times Z$ is trivial on the first factor. Indeed, for all $g \in F$, $u \in P_u$ and $x \in Z$ denote by $g \cdot (u, x) = (\rho_1(g)(u), \rho_2(g)(x))$ the transformation of (u, x) by g induced by the bijection π_\star . Then we have

$$u \cdot (g \cdot x) = g \cdot (u \cdot x) = g \cdot \pi_\star(u, x) = \pi_\star(\rho_1(g)(u), \rho_2(g)(x)) = \rho_1(g)(u) \cdot \rho_2(g)(x).$$

Hence $\rho_1(g)(u) = u$ and $\rho_2(g)(x) = g \cdot x$. This finishes the proof of the theorem. \square

4.2. Parametrization of G -divisors. For a normal G -variety X with spherical orbits, our next task is to give a parameterization of the G -divisors of X in terms of its defining colored fan \mathcal{E} (see Theorem 4.6). Note that in the case where X is a toric \mathbb{T} -variety and \mathcal{E} is a usual fan, the \mathbb{T} -divisors of X are naturally in bijection with the one-dimensional cones of \mathcal{E} . The idea is to have a similar description in this context. We start with the following technical lemma.

Lemma 4.3. *Let $X = X(\mathcal{D}, \mathcal{F}, \gamma)$ be a simple normal G -variety with spherical orbits. Let us consider the B -chart $X'_0 \subseteq X$ attached to $(\mathcal{D}, \mathcal{F}, \gamma)$, let Z be a G -cycle in X and denote by ν the G -valuation centered in its generic point. Then the intersection $Z' = Z \cap X'_0$ is identified via the map π_\star with the product $P_u \times Z_\star$, where Z_\star is a G_\star -cycle in X'_\star with corresponding G_\star -valuation $\nu|_{k(X'_\star)}$. Furthermore, the map $Z \mapsto Z_\star$ induces an injective map between the set of G -cycles in X and the set of G_\star -cycles in X'_\star which preserves the inclusion order. It moreover induces a bijection between the subset of G -cycles in X and the subset of G_\star -cycles which corresponds to non-central invariant valuations under the groups G and G_\star , respectively.*

Proof. The vanishing ideal I of Z' is generated by the functions $f \in k[X'_0] \setminus \{0\}$ such that $\nu(f) > 0$. Denote by Z_1 the G_\star -cycle of X'_\star defined by the ideal

$$I' = \{f \in k[X'_\star] \setminus \{0\} \mid \nu(f) > 0\} \cup \{0\}.$$

The vanishing ideal of $P_u \times Z_1$ is $k[P_u] \otimes_k I'$ which is contained in I . Hence $Z' \subseteq P_u \times Z_1$. Moreover,

$$I = k[P_u] \otimes_k I(Z_\star) \subseteq k[P_u] \otimes_k I',$$

where $I(Z_\star) \subseteq k[X'_\star]$ is the vanishing ideal of Z_\star . This gives the equality $Z' = P_u \times Z_1$ and so $Z_\star = Z_1$ by taking the quotient by P_u . Now it is clear that Z_\star is determined by $\nu_\star := \nu|_{k(X'_\star)}$ and that $Z \mapsto Z_\star$ is injective (since ν and ν_\star are represented by the same element in \mathcal{Q}).

Let us show that the map is a bijection in the context of non-central invariant valuations. We construct an inverse map ϕ as follows. Let Z_\star be a G_\star -cycle of X'_\star and denote by ν_\star the G_\star -valuation with center the generic point of Z_\star . Then ν_\star is represented by a class $[s, p, \ell]$, where $s \in \varsigma(S)$, $\ell \in \mathbb{Q}_{>0}$ and p belongs to the dual of cone generated by $\Sigma \cap \text{vect}_{\mathbb{Q}}(\Delta_\star)$. Changing \mathcal{D} by the pull back of a projective birational morphism, we may assume that $s = v_Y$ for some prime divisor $Y \subseteq \Gamma$. Hence $p/\ell \in \mathcal{D}_Y$ and therefore $[s, p, \ell] \in \mathcal{Q}_\Sigma$. Now we can take the G -valuation ν represented by this class set. Denote by $Z \subseteq X$ the G -cycle associated with ν (see Lemma 2.20). Then we claim that the assignment $\phi(Z_\star) = Z$ does not depend on the choice of ν_\star . Indeed, according to the previous argument we have $Z \cap X'_0 = P_u \times Z_\star$. This finishes the proof of the lemma. \square

Remark 4.4. The map $Z \mapsto Z_\star$ is not a bijection in general. As explained in [Tim11, Remark 15.19], a color of X could be contracted onto a G_\star -cycle (by observing that \mathcal{V} could be strictly included in \mathcal{V}_\star).

As a tool to study the geometry of normal G -varieties with spherical orbits, we introduce the *contraction morphism*. With the same notation as in Lemma 4.3, we consider an affine F -stable open covering $(U_i)_{i \in I}$ of the projective smooth variety Γ . Let \mathcal{E}_c be the colored divisorial fan generated by $\{(\mathcal{D}|_{U_i}, \mathcal{F}) \mid i \in I\}$. Note that the inclusions $C(\mathcal{D}|_{U_i}) \subseteq C(\mathcal{D})$ induce a natural G -morphism

$$\pi_c : X(\mathcal{E}_c, \gamma) \rightarrow X = X(\mathcal{D}, \mathcal{F}, \gamma).$$

The next result collects some properties about the contraction map π_c . The reader is referred to [AH06, Theorem 3.1] for the case of torus actions.

Proposition 4.5. *The G -morphism π_c is proper birational and does not depend on the choice the open covering $(U_i)_{i \in I}$. Moreover, π_c is a resolution of the indeterminacy locus of the rational quotient $X \dashrightarrow \Gamma$.*

Proof. To check the properness of $X(\mathcal{E}_c) \rightarrow X$ we only need to show that if ν is any G -valuation of the function field $k(X)$ with center in X , then ν has a center in $X(\mathcal{E}_c)$ (see [Tim11, Theorem 12.13]). This directly follows from Lemma 2.20. As a result, we obtain a commutative diagram

$$\begin{array}{ccc} X(\mathcal{E}_c) & \xrightarrow{\quad} & X \\ \downarrow /F & & \downarrow /F \\ X(\mathcal{E}_c, \gamma) & \xrightarrow{\pi_c} & X' \end{array}$$

The vertical maps are proper since they are quotient maps by F . Hence the morphism π_c is proper. Remark that the complement of the union of colors that does not belong to \mathcal{F} is identified to $P \times^{G_\star} \tilde{X}_\star = P_u \times \tilde{X}_\star$ (see Theorem 4.2), where \tilde{X}_\star is the relative spectrum of

$$\mathcal{A}_\star = \bigoplus_{\lambda \in \text{Tail}(\mathcal{D})^\vee \cap M} \mathcal{O}_\Gamma(\mathcal{D}(\lambda)) \otimes_k V_\lambda.$$

Therefore the rational quotient of $X(\mathcal{E}_c)$ by G is a morphism since it is induced by the natural morphisms $\text{Spec } \mathcal{A}_\star \rightarrow \text{Loc}(\mathcal{D})$. This finishes the proof of the proposition. \square

We now introduce the set of G -valuations for describing the G -divisors of a normal G -variety with spherical orbits.

Vertical valuations. Let us start with a single colored polyhedral divisor $(\mathcal{D}, \mathcal{F})$. We denote by $\text{Vert}(\mathcal{D})$ the set of pairs $([Y], v)$, where $[Y]$ is an F -orbit of a prime divisor of $\text{Loc}(\mathcal{D})$ and v is vertex of \mathcal{D}_Y such that the following conditions hold. The subset $\Gamma(\text{Loc}(\mathcal{D}), \mathcal{O}(-Y) \cdot \mathcal{A})$ is the ideal of a prime divisor of $\text{Spec } A(\text{Loc}(\mathcal{D}), \mathcal{D})$, where we recall that

$$\mathcal{A} = \bigoplus_{m \in \text{Tail}(\mathcal{D})^\vee \cap M} \mathcal{O}(\mathcal{D}(m)).$$

Note that this notion does not depend on the choice of a representative $Y \in [Y]$. For an element of $([Y], v) \in \text{Vert}(\mathcal{D})$, the center of the G -valuation $[v_{[Y]}, \mu(v)v, \mu(v)]$ of $k(\mathcal{X}')$ is the generic point of a G -cycle $D_{[Y],v} \subseteq X(\mathcal{D}, \mathcal{F}, \gamma)$ (see Lemma 2.20), where $\mu(v)$ is the smallest integer $l \in \mathbb{Z}_{>0}$ such that $l \cdot v \in N$.

Horizontal valuations. For simplicity we write by the same letter a ray (i.e. a one-dimensional face) of a polyhedral cone and its primitive lattice vector. We will confuse these two notions when it is needed. We denote by $\text{Ray}(\mathcal{D}, \mathcal{F})$ the set of rays ρ of $\sigma = \text{Tail}(\mathcal{D})$ such that $\varrho(\mathcal{F}) \cap \rho = \emptyset$ and $\mathcal{D}(m)$ is a big Cartier \mathbb{Q} -divisor for any m in the relative interior of $\sigma^\vee \cap \rho^\perp$. Similarly, for an element of $\rho \in \text{Ray}(\mathcal{D}, \mathcal{F})$, the center of the G -valuation $[\cdot, \rho, 0]$ of $k(\mathcal{X}')$ is the generic point of a G -cycle $D_\rho \subseteq X(\mathcal{D}, \mathcal{F}, \gamma)$ (see Lemma 2.20). We define more generally the sets

$$\text{Vert}(\mathcal{E}) = \bigcup_{(\mathcal{D}, \mathcal{F}) \in \mathcal{E}} \text{Vert}(\mathcal{D}) \text{ and } \text{Ray}(\mathcal{E}) = \bigcup_{(\mathcal{D}, \mathcal{F}) \in \mathcal{E}} \text{Ray}(\mathcal{D}, \mathcal{F})$$

for an arbitrary colored divisorial fan \mathcal{E} on $(\Gamma, \mathcal{S}, \gamma)$.

In the next result, we use the contraction morphism defined above and the local structure theorem to parametrize the set of G -divisors of a normal G -variety X with spherical orbits. We also give a presentation by generators and relations of the class of group of X . We refer to [FZ03, Theorem 4.22], [PS11, Corollary 3.15], [LT16, Corollary 2.12] for special cases where the theorem was proven.

Theorem 4.6. *Let X be a normal G -variety with defining colored divisorial fan \mathcal{E} on $(\Gamma, \mathcal{S}, \gamma)$. Let $\text{Div}(\mathcal{E})$ be the set of G -divisors of X . Then the map*

$$\phi : \text{Vert}(\mathcal{E}) \bigsqcup \text{Ray}(\mathcal{E}) \rightarrow \text{Div}(\mathcal{E}), ([Y], v) \mapsto D_{[Y],v}, \rho \mapsto D_\rho.$$

is well-defined and bijective. Moreover, the class group $\text{Cl}(X)$ is isomorphic to the abelian group

$$\text{Cl}(\text{Loc}(\mathcal{E})/F) \oplus \bigoplus_{([Y],v) \in \text{Vert}(\mathcal{E})} \mathbb{Z}D_{[Y],v} \oplus \bigoplus_{\rho \in \text{Ray}(\mathcal{E})} \mathbb{Z}D_\rho \oplus \bigoplus_{D \in \mathcal{F}_\Omega} \mathbb{Z}D,$$

modulo the relations

$$[Y] = \sum_{(y,v) \in \text{Vert}(\mathcal{E})} \mu(v) D_{[Y],v} \text{ and}$$

$$\sum_{([Y],v) \in \text{Vert}(\mathcal{E})} \mu(v) \langle m, v \rangle D_{[Y],v} + \sum_{\rho \in \text{Ray}(\mathcal{E})} \langle m, \rho \rangle D_\rho + \sum_{D \in \mathcal{F}_\Omega} \langle m, \varrho(D) \rangle D = 0,$$

where $m \in M$ and $Y \subseteq \text{Loc}(\mathcal{E})$ is a prime divisor.

Proof. Without loss of generality, we may assume that $X = X(\mathcal{D}, \mathcal{F}, \gamma)$ is simple. Since X is a separated scheme over k , the map ϕ is injective if it is well-defined. Let Z be a G -divisor of X . Then Z is the image under π_c of a G -divisor \tilde{Z} of $\tilde{X} := X(\mathcal{E}_c, \gamma)$ represented by the same valuation ν .

Case 1. Assume that \tilde{Z} is sent dominantly on $\text{Loc}(\mathcal{E})/F$, or equivalently that ν is central, i.e., ν is trivial on $k(S)^F$. Moreover, $\nu = [\cdot, \rho, 0]$ for some $\rho \in N \cap \mathcal{V}$. Hence \tilde{Z} lifts uniquely into a G -divisor on $X(\mathcal{E}_c)$ that we denote by the same letter. Let S_0 be the complement in $\text{Loc}(\mathcal{D})$ of all prime divisors where the polyhedral coefficients of \mathcal{D} are non-trivial. Remark that $S_0 \times X_{\sigma, \mathcal{F}} \subseteq X(\mathcal{E}_c)$ is a G -stable dense open subset, where $X_{\sigma, \mathcal{F}}$ is the spherical G -variety which is the general fiber of the quotient map $q : X(\mathcal{E}_c) \rightarrow \text{Loc}(\mathcal{D})$, the pair (σ, \mathcal{F}) is the colored cone of $X_{\sigma, \mathcal{F}}$ and σ is the tail of \mathcal{D} . In addition, we have $\tilde{Z} \cap (S_0 \times X_{\sigma, \mathcal{F}}) \neq \emptyset$. Thus by using [Kno91, Lemma 2.4] and Lemma 2.20, we conclude that $\rho \in \text{Ray}(\mathcal{E}_c)$.

Again, we write by the letter Z the lift of Z under the quotient map by F . Let us consider the sheaf of $\mathcal{O}_{\text{Loc}(\mathcal{D})}$ -algebras

$$\mathcal{A}_\rho := \bigoplus_{\lambda \in \sigma^\vee \cap \rho^\perp \cap M} \mathcal{O}(\mathcal{D}(\lambda)) \otimes V_\lambda$$

such that $Z \cap X_0 = P_u \times \text{Spec } \Gamma(\text{Loc}(\mathcal{D}), \mathcal{A}_\rho)$ for some B -chart $X_0 \subseteq X$ intersecting Z (see Lemma 4.3). Now $\mathcal{D}(\lambda)$ is big on the relative interior of $\sigma^\vee \cap \rho^\perp \cap M$ if and only if

$$\dim Z = \dim \Gamma(\text{Loc}(\mathcal{D}), \mathcal{A}_\rho) = \dim \text{Spec}_{\text{Loc}(\mathcal{D})} \mathcal{A}_\rho = \dim \tilde{Z}$$

if and only if $\dim X - \dim Z = \dim \tilde{X} - \dim \tilde{Z} = 1$. We finally conclude that $\rho \in \text{Ray}(\mathcal{E})$ and $Z = D_\rho$. Our analysis shows that the assignment $\rho \mapsto D_\rho$ is a well-defined map.

Case 2. We now pass to the case where the image of \tilde{Z} by q is not dense. Since \tilde{Z} is a prime divisor, the closure Y of $q(\tilde{Z})$ in $\text{Loc}(\mathcal{D})$ is a codimension one subset. In addition, $\nu = [v_Y, p, \ell] \in \mathcal{Q}_\Sigma$ with $\ell \neq 0$. Let $Z_1 \subseteq X(\mathcal{E}_c)$ be the G -stable subvariety obtained as the union of G -orbits contained in

$$\bigcup_{D \in \mathcal{F}_\Omega} D \text{ and set } X_1 = X(\mathcal{E}_c) \setminus Z_1.$$

Since the codimension of each irreducible component of Z_1 is at least 2, we have $X_1 \cap \tilde{Z} \neq \emptyset$. Then using the proof of Theorem 2.27, X_1 is described by an uncolored colored divisorial fan on (Γ, \mathcal{S}) that we can suppose to be a singleton $\{(\mathcal{D}_1, \emptyset)\}$. We see $(\mathcal{D}_1, \emptyset)$ as an element of \mathcal{E}_c . Now by virtue of Theorem 4.2, the B -chart $X_0(\mathcal{D}_1, \emptyset)$ is expressed as a product $P'_u \times \text{Spec } A(\text{Loc}(\mathcal{D}_1), \mathcal{D}_1)$. Denoting by

$$q_1 : \text{Spec } A(\text{Loc}(\mathcal{D}_1), \mathcal{D}_1) \rightarrow \text{Loc}(\mathcal{D}_1)$$

the quotient map, the G -divisor \tilde{Z} is equal to the closure of $P'_u \times Z_2$, where Z_2 is an irreducible component of $q_1^{-1}(q(\tilde{Z}) \cap \text{Loc}(\mathcal{D}_1))$. According to [PS11, Proposition 3.13] we deduce that $(p, \ell) = (\mu(v)v, \mu(v))$ for some vertex v of a polyhedral coefficient of \mathcal{D}_1 and so $([Y], v) \in \text{Vert}(\mathcal{E}_c)$.

We denote by the same letter Z a lift of Z under the quotient map by F . Let us consider the sheaf of $\mathcal{O}_{\text{Loc}(\mathcal{D})}$ -algebras

$$\mathcal{A}_{Y,v} := \bigoplus_{\lambda \in \sigma^\vee \cap M} \mathcal{O}([\mathcal{D}(\lambda)] - Y) \otimes V_\lambda$$

such that $Z \cap X'_0 = P''_u \times \text{Spec } \Gamma(\text{Loc}(\mathcal{D}), \mathcal{A}_{Y,v})$ for some B -chart $X'_0 \subseteq X$ intersecting Z (see Lemma 4.3). Now $(v, [Y]) \in \text{Vert}(\mathcal{E})$ (adapt arguments of the proof of [HS10, Proposition 4.11]) if and only if

$$\dim Z = \dim \Gamma(\text{Loc}(\mathcal{D}), \mathcal{A}_{Y,v}) = \dim \text{Spec}_{\text{Loc}(\mathcal{D})} \mathcal{A}_{Y,v} = \dim \tilde{Z},$$

if and only if Z is of codimension one. Our analysis shows that the assignment ϕ is a well-defined bijective map.

For the presentation of the class group $\text{Cl}(X)$ by generators and relations, the proof is based on the same argument as in [PS11, Corollary 3.15], [LT16, Corollary 2.12]. We recall the key argument. By [FMSS95] every divisor of X is linearly equivalent to a B -stable divisor. Hence $\text{Cl}(X)$ is the quotient of the free abelian group of the B -stable divisors modulo the subgroup of principal divisors associated with the B -eigenfunctions $f \otimes \chi^m$ of $k(X)$, where $f \in k(Y)^*$ and χ^m is as in 2.1. The calculation of $\text{Cl}(X)$ follows from the expression of $\text{div}(f \otimes \chi^m)$ in terms of B -valuations of $k(X)$ corresponding to B -divisors. This finishes the proof of the theorem. \square

5. CANONICAL CLASS

In this section, we investigate the canonical class of a normal G -variety with spherical orbits. We will denote by \sim for the linear equivalence relation between Weil divisors. Our main result can be stated as follows (see [PS11, Theorem 3.21] for the special case of normal \mathbb{T} -varieties).

Theorem 5.1. *Let K_X be a canonical divisor of a normal G -variety X with spherical orbits defined by a colored divisorial fan \mathcal{E} on $(\Gamma, \mathcal{S}, \gamma)$. Then we have*

$$\sharp F \cdot K_X \sim -\sharp F \left(\sum_{\rho \in \text{Ray}(\mathcal{E})} D_\rho + \sum_{D \in \mathcal{F}_X} a_D D - \sum_{([Y], v) \in \text{Vert}(\mathcal{E})} \left[\frac{\mu(v)}{r_{[Y], v}} b_{[Y]} + \mu(v) - 1 \right] D_{[Y], v} \right),$$

where $K_{\text{Loc}(\mathcal{E})} = \sum_{Y \subseteq \text{Loc}(\mathcal{E})} b_Y \cdot Y$ is a canonical divisor of the variety $\text{Loc}(\mathcal{E})$, the number $b_{[Y]}$ stands for $\sum_{Y \in [Y]} b_Y$, \mathcal{F}_X is the set of colors of X and $a_D \in \mathbb{Z}_{\geq 1}$ for any $D \in \mathcal{F}_X$. Moreover, the ramification index $r_{[Y], v}$ is defined as

$$r_{[Y], v} = \sharp \{g \in F \mid g \cdot D_{Y, v} \subseteq D_{Y, v} \text{ and } g \cdot x = x\},$$

where $D_{Y, v} \subseteq \gamma^{-1}(D_{[Y], v})$ is an irreducible component and x is a generic point of $D_{Y, v}$. This number does not depend on the choice of the prime divisor $D_{Y, v}$.

Proof. We separate the proof into two parts. In the first part, we determine the canonical class of X in the case where X is a G -model of $\mathcal{X} = S \times \Omega$. In the second part, we deduce the general case to the first step by applying the Riemann-Hurwitz formula for the quotient map $\gamma : X(\mathcal{E}) \rightarrow X(\mathcal{E}, \gamma)$.

Case 1. Assume that X is a G -model of \mathcal{X} . We first make some reduction by taking a local chart and removing closed subsets of codimension ≥ 2 . In this way, we may suppose that X is smooth and X is determined by an uncolored colored polyhedral divisor (\mathcal{D}, \emptyset) with smooth affine locus. We will consider the two following open subsets of the G -variety X : (1) the subset $X_1 = S_0 \times X_\Theta$ where we remove the special fibers of the quotient map by G . The G -variety X_Θ is the simple spherical embedding of Ω with colored cone $\Theta = (\text{Tail}(\mathcal{D}), \emptyset)$ which plays the role of the generic fiber. In addition, S_0 is an open subset of $\text{Loc}(\mathcal{D})$; (2) The B -chart $X_0 \subseteq X$ associated with (\mathcal{D}, \emptyset) which is the complement of the union of the colors of X . By Theorem 4.2, the B -chart X_0 is identified with a product $P_u \times \text{Spec } A(\text{Loc}(\mathcal{D}), \mathcal{D})$, where P_u is an affine space. Note that the complement of $X_0 \cup X_1$ in X is a closed subset with irreducible components of codimension at least 2.

Let α be the exterior product of a basis of the sheaf of module differential forms of P_u and let δ be a global section of the canonical bundle of $\text{Loc}(\mathcal{D})$. For a basis e_1, \dots, e_n of M , let $\chi^{e_1}, \dots, \chi^{e_n}$ be the associated Laurent monomials. Then by the argument of the proofs of [Bri97, Section 4.1, Proposition 4.1] and [PS11, Theorem 3.21], the differential form

$$\omega = \alpha \wedge \delta \wedge \frac{d\chi^{e_1}}{\chi^{e_1}} \wedge \dots \wedge \frac{d\chi^{e_n}}{\chi^{e_n}}$$

restricts on X_0 , X_1 and $X_0 \cap X_1$ to generators of the canonical sheaves on X_0 , X_1 and $X_0 \cap X_1$. Consequently, the differential form ω is a global section of the canonical bundle of X . Hence for computing a canonical divisor K_X , it is sufficient to determine the order $v_D(\omega)$ of ω along any prime divisor D of X . Restricting on the chart X_1 , we have $v_D(\omega) \in \mathbb{Z}_{<0}$ and $v_{D_\rho}(\omega) = -1$ for all $\rho \in \text{Ray}(\mathcal{E})$ and $D \in \mathcal{F}_X$ (compare with [Bri97, Section 4.1, Proposition 4.1]). Restricting on X_0 , the computation of the order of ω at $D_{Y, v}$ for $(Y, v) \in \text{Vert}(\mathcal{E})$ follows from the argument of the proof of [PS11, Theorem 3.21], where the notation $\text{Vert}(\mathcal{E})$ is considered for the trivial F -action. Furthermore, the order of ω along a prime divisor which is not B -stable is equal to 0. We finally obtain the formula

$$K_X = - \sum_{\rho \in \text{Ray}(\mathcal{E})} D_\rho + \sum_{(Y, v) \in \text{Vert}(\mathcal{E})} (\mu(v)(b_Y + 1) - 1) D_{Y, v} - \sum_{D \in \mathcal{F}_X} a_D D,$$

where $K_{\text{Loc}(\mathcal{D})} = \sum_{Y \subseteq \text{Loc}(\mathcal{D})} b_Y \cdot Y$ is a canonical divisor of $\text{Loc}(\mathcal{D})$ and $a_D \in \mathbb{Z}_{\geq 1}$ for any $D \in \mathcal{F}_X$.

Case 2. Let us assume that X is a G -model of $\mathcal{X}' = \mathcal{X}/F$ with colored divisorial fan \mathcal{E} defined on $(\Gamma, \mathcal{S}, \gamma)$. Consider the quotient map $\gamma : X(\mathcal{E}) \rightarrow X$ by F . By the Riemann-Hurwitz formula, we have $K_{X(\mathcal{E})} \sim \gamma^* K_X + R$, where $R = \sum_{i \in I} (r_i - 1) R_i$ is a divisor supported on the ramification locus of γ and r_i is precisely the ramification index attached to the prime divisor R_i . We recall that the ramification

locus is the smallest closed subset $Z \subseteq X(\mathcal{E})$ such that γ is étale on $X(\mathcal{E}) \setminus Z$ and so R is G -stable. Now using [Liu02, Theorem 2.18, page 271] we obtain that

$$\sharp F \cdot K_X = \gamma_* \gamma^* K_X \sim \gamma_* K_{X(\mathcal{E})} - \gamma_* R.$$

Claim. The F -action is free on the general points of the divisors D_ρ and $D \in \mathcal{F}_{X(\mathcal{E})}$. Indeed, by making the same reduction as in Case 1, we may find a $G \times F$ -stable dense open subset of the form $S_0 \times X_\Theta$ such that the F -action is free. We conclude the claim by remarking that the open subset $S_0 \times \Omega$ intersects any divisor D_ρ and any color $D \in \mathcal{F}_{X(\mathcal{E})}$.

So the ramification divisor R is determined by a finite number of prime divisors corresponding to elements of the set $\text{Vert}(\mathcal{E})$. By a direct computation we have

$$\gamma_* R = \sum_{([Y], v) \in \text{Vert}(\mathcal{E})} \left(\sum_{D_{Y,v} \subseteq \gamma^{-1}(D_{[Y],v})} [k(D_{Y,v}) : \gamma^* k(D_{[Y],v})] (r_{Y,v} - 1) \right) D_{[Y],v},$$

where $r_{Y,v}$ is the ramification index of $D_{Y,v}$ and $k(D_{[Y],v})$, $k(D_{Y,v})$ are the residue fields of the generic points of $D_{[Y],v}$ and $D_{Y,v}$, respectively. Since the F -action permutes transitively the irreducible components of $\gamma^{-1}(D_{[Y],v})$ for any $([Y], v) \in \text{Vert}(\mathcal{E})$, we have $r_{Y,v} = r_{Y',v}$ and

$$[k(D_{Y,v}) : \gamma^* k(D_{[Y],v})] = [k(D_{Y',v}) : \gamma^* k(D_{[Y],v})] \text{ for all prime divisors } D_{Y,v}, D_{Y',v} \text{ in } \gamma^{-1}(D_{[Y],v}).$$

Moreover, the formula involving ramification indices and inertial degrees gives

$$\sharp F = [k(X(\mathcal{E})) : \gamma^* k(X)] = \sum_{D_{Y,v} \subseteq \gamma^{-1}(D_{[Y],v})} [k(D_{Y,v}) : \gamma^* k(D_{[Y],v})] r_{Y,v}.$$

Let us translate this formula in term of the F -action on $X(\mathcal{E})$. We denote by $F_{Y,v}$ the subgroup of F which preserves $D_{Y,v}$. For x generic in $D_{Y,v}$ we write by $\text{Stab } D_{Y,v}$ its stabilizer for the action $F_{Y,v}$ on $D_{Y,v}$. Then using Lemma 3.2 (iv) we have

$$[k(D_{Y,v}) : \gamma^* k(D_{[Y],v})] = [k(D_{Y,v}) : k(D_{Y,v})^{F_{Y,v}/\text{Stab } D_{Y,v}}] = \frac{\sharp F_{Y,v}}{\sharp \text{Stab } D_{Y,v}} \text{ and } \sharp[Y] = \frac{\sharp F}{\sharp F_{Y,v}}.$$

Hence the equality $\sharp F = \sharp[Y] \cdot [k(D_{Y,v}) : \gamma^* k(D_{[Y],v})] \cdot r_{Y,v}$ by the argument above yields $r_{Y,v} = \sharp \text{Stab } D_{Y,v}$, as required. Denoting by $r_{[Y],v}$ for $r_{Y,v}$ we finally arrive to

$$\gamma_* R = \sum_{([Y], v) \in \text{Vert}(\mathcal{E})} (\sharp F - \sharp F / r_{[Y],v}) D_{[Y],v}.$$

By Case 1, it follows that $\gamma_* K_{X(\mathcal{E})}$ is equal to

$$-\sharp F \left(\sum_{\rho \in \text{Ray}(\mathcal{E})} D_\rho + \sum_{D \in \mathcal{F}_X} a_D D \right) + \sum_{([Y], v) \in \text{Vert}(\mathcal{E})} \left(\mu(v) \frac{\sharp F}{r_{[Y],v}} (b_{[Y]} + 1) - \frac{\sharp F}{r_{[Y],v}} \right) D_{[Y],v},$$

where for any $D \in \mathcal{F}_X$ we let $a_D = \sum_{D' \subseteq \gamma^{-1}(D)} a_{D'}$. The difference $\gamma_* K_{X(\mathcal{E})} - \gamma_* R$ and the preceding computations give the desired formula. \square

Remark 5.2. The coefficient a_D in Theorem 5.1 can be explicitly determined in term of the homogeneous spherical datum \mathcal{S} of Ω . We refer to [Bri97, Theorem 4.2] for more details.

The next proposition gives an example where there are no ramification divisors.

Proposition 5.3. *With the same notation as in Theorem 5.1, if the group F acts freely on Γ , then $r_{[Y],v} = 1$ for all $([Y], v) \in \text{Vert}(\mathcal{E})$.*

Proof. Note that under our assumption the F -action on the variety $X(\mathcal{E}_c)$ is free. Indeed, if $g \in F$ belongs to the stabilizer of a point $x \in X(\mathcal{E}_c)$, then we have $g \cdot \pi(x) = \pi(g \cdot x) = \pi(x)$, where $\pi : X(\mathcal{E}_c) \rightarrow \Gamma$ is the quotient map by G . Since the F -action on Γ is free, we deduce that $g = 1$, as required. Now there exists a $(G \times F)$ -stable dense open subset of $X(\mathcal{E})$ with complement of codimension ≥ 2 and $(G \times F)$ -isomorphic to the one of $X(\mathcal{E}_c)$. Hence the splitting $\gamma : X(\mathcal{E}) \rightarrow X$ has no ramification divisor and so $r_{[Y],v} = 1$ for all $([Y], v) \in \text{Vert}(\mathcal{E})$. \square

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MATHEMATISCHES INSTITUT, HEINRICH HEINE UNIVERSIT T, 40225 D SSELDORF, GERMANY.

E-mail address: langlois.kevin18@gmail.com